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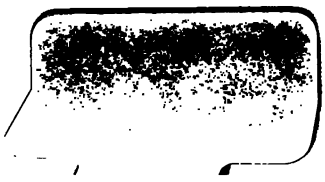

MATHEMATICAL

EXAMINATION PAPERS.

MILITARY EXAMINATIONS.

MATHEMATICAL

EXAMINATION PAPERS,

SET FOR

ENTRANCE TO R.M.A., WOOLWICH,

WITH ANSWERS.

BY

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PREFACE.

THE following papers, set at recent examinations for entrance to Woolwich Academy, have been issued, and Answers to the Examples annexed, from a feeling that such a publication would be useful to Military Tutors and Pupils, as well as to a large and increasing class of persons engaged in acquiring or imparting a knowledge of Pure and Mixed Mathematics. Much care has been bestowed upon the working out of the results; but the compiler cannot hope that his book is free from errors, and he will therefore feel very thankful for any corrections that may be sent to him.

ROCHESTER HOUSE, EALING, W.,
April, 1880.

SET I.

EUCLID.

1. Any two sides of a triangle are together greater than the third side.

   If D be a point on the base BC of the triangle ABC, prove that AD is less than the greater of the two sides AB and AC, and if E be a point on either of these two sides, prove that DE is less than the greatest side of the triangle.

2. The complements of the parallelograms which are about the diameter of any parallelogram, are equal to one another.

3. If a straight line be divided into two equal parts and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.

4. Divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts may be equal to the square on the other part.

5. The angle at the centre of a circle is double of the angle at the circumference on the same base, that is, on the same arc.

   A triangle ABC is inscribed in a circle of which O is the centre and the arc BC is bisected at D.

   Prove that the angle ADO is half the difference of the angles ABC and ACB.

6. If from any point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it, and if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square on the line which meets the circle, the line which meets the circle shall touch it.

7. Describe a circle about a given triangle.

   Given the radius of the circumscribing circle, the length of one of the sides, and the area of the triangle, construct the triangle.

8. Inscribe an equilateral and equiangular quindecagon in a given circle.

9. The sides about the equal angles of triangles which are equiangular to one another are proportionals; and those which are opposite to the equal angles are homologous sides, that is, are the antecedents or the consequents of the ratios.

The chords AB and CD in the circle ABCD are produced towards B and D respectively to meet in the point E, and through E, the line EF is drawn parallel to AD to meet CB produced in F.

Prove that EF is a mean proportional between FB and FC.

10. If four straight lines be proportionals the similar rectilinear figures similarly described upon them shall be proportionals.

ALGEBRA.

1. When (m) and (n) are whole numbers, express the result of the division of a^m by a^n; (1) when (m) is greater than (n); (2) when (m) is less than (n), and trace the steps in the theory of indices from which it is inferred that a^{-n} represents $\frac{1}{a^n}$, and that $a^{\frac{1}{n}}$ represents $\sqrt[n]{a}$.

   Express by an arithmetical fraction $(8^{\frac{2}{3}} + 4^{\frac{1}{2}}) \times 16^{-\frac{3}{4}}$.

2. Multiply $(x^{\frac{3}{2}} + 2y^{\frac{3}{2}} + 3z^{\frac{3}{2}})$ by $(x^{\frac{3}{2}} - 2y^{\frac{3}{2}} - 3z^{\frac{3}{2}})$.

3. Divide $(x+y)^2 + (x+z)^2 + (y+z)^2 + 2(x+y)(x+z) + 2(x+y)(y+z) + 2(x+z)(y+z)$ by $(x+y+z)$.

4. Express $a^2(c-b) + b^2(a-c) + c^2(b-a)$ as the product of three simple binomial factors.

5. Reduce to its lowest terms
$$\frac{(a^2-b^2)(x^2-y^2) - 4abxy}{(a^2-b^2)(x^2+y^2) + 2(a^2+b^2)xy}$$

6. Prove
$$\frac{a}{(a-b)(c-a)(x+a)} + \frac{b}{(a-b)(b-c)(x+b)} + \frac{c}{(b-c)(c-a)(x+c)} = \frac{x}{(x+a)(x+b)(x+c)}$$

7. Show how to find a factor which will rationalise the surd quantity $a^{\frac{1}{p}} + b^{\frac{1}{q}}$, where (p) and (q) are any whole numbers.

   Express $\dfrac{5^{\frac{1}{2}} - 7^{\frac{1}{4}}}{5^{\frac{1}{2}} + 7^{\frac{1}{4}}}$ by an equivalent fraction with a rational denominator.

8. Solve the following equations :—

   (1.) $(x-a)(x-b) = (x-c)(x-d)$;

   (2.)
$$\left.\begin{aligned} \frac{3}{x} + \frac{5}{y} &= \frac{8}{15} \\ 9y - 22x &= \frac{3xy}{25} \end{aligned}\right\}$$

   (3.) $\dfrac{x^2}{6} - \dfrac{1}{7} = \dfrac{43x - 8x^2}{42}$;

   (4.)
$$\left.\begin{aligned} x^2(x+y+z) &= 108 \\ y^2(x+y+z) &= 192 \\ z^2(x+y+z) &= 300 \end{aligned}\right\}$$

9. What are the roots of a quadratic equation? If (α) and (β) are the roots of the equation $x^2 - px + q = 0$, and (α^4) (β^4) the roots of $x^2 - Px + Q = 0$; express P and Q in terms of (p) and (q).

10. Two workmen (A) and (B) of unequal efficiency, working together are capable of making a drain, which they undertake to do, in 60 days. After working for 20 days A falls ill, and his place is supplied by another workman (C), whose efficiency is the same as that of B, and it is found that the whole time consumed in making the drain is 80 days. In how many days would A or B working alone make the drain?

11. A cubical tank contains 512 cubic feet of water. It was required to enlarge the tank, the depth remaining the same, so that it should contain seven times as much water as before, subject to the condition that the length added to one side of the base should be four times the length added to the other side. Find the sides of the new rectangular base.

12. Define the three progressions, Arithmetical, Geometrical, Harmonical; and show that the three equations following severally determine for the three quantities (a) (b) (c) the conditions of each progression:

(1.) $\frac{a-b}{b-c} = \frac{a}{a}$; (2.) $\frac{a-b}{b-c} = \frac{a}{b}$; (3.) $\frac{a-b}{b-c} = \frac{a}{c}$.

Insert four harmonic means between $\frac{1}{3}$ and 2.

A watch, which is set right at noon, gains two minutes the first hour afterwards, three the second, four the third, and so on; after what interval will the watch be an hour and a half fast, and what time will it then indicate?

13. Assuming the expression for the number of permutations of (n) things taken (r) together, find the expression for the number of combinations of (n) things taken (r) together.

A dealer has for sale 8 bay, 7 grey, and 5 black horses. A purchaser requests that 12 horses, four of each colour, may be sent him; in how many different ways can the dealer execute the order?

14. Assuming the form of the expansion of a binomial when the index is a positive integer, and that $f(m)$ denotes the series $1 + mx + \frac{m(m-1)}{1.2}x^2 +$ &c. for any value of (m); show how to obtain a proof of the theorem when the index is a negative integer.

Since $(1-x)^n(1-x)^{-n} = 1$, it would follow that if the two expanded series for $(1-x)^n$ and $(1-x)^{-n}$ be multiplied together all the coefficients of the different powers of (x) should vanish separately; show that this is the case for the whole coefficient of x^4.

PLANE TRIGONOMETRY.

1. What in plane geometry is the limit of the magnitude of an angle? In what sense in analytical Trigonometry is the magnitude of an angle considered unlimited, and either positive or negative? Which of the ordinary trigonometrical functions are limited in magnitude and which unlimited?

   If $\sin A = a$, express by a general formula all the angles which have (a) for their sine; if $\cos A = b$, prove $a^2 + b^2 = 1$.

2. Find sin A and cos A in terms respectively of the sines and cosines of their supplements. Through what portions of the first four quadrants is the sine greater than the cosine in magnitude, and the tangent greater than the cotangent; when are they respectively equal and of the same sign?

   Find sin 120°, tan 60°, cos 135°.

   Prove $\sin A = \dfrac{\tan A}{\sqrt{1 + (\tan A)^2}}$.

3. Prove $\cos (A + B) = \cos A \cos B - \sin A \sin B$, and deduce from the formula the corresponding expressions for $\sin (A + B)$, $\sin (A - B)$, and $\cos 2A$.

4. Prove—

   (1.) $\dfrac{\text{Sin A} + \sin B}{\text{Cos A} + \cos B} = \tan \dfrac{A + B}{2}$.

   (2.) $\text{Cos } 72° = \dfrac{\sqrt{5} - 1}{4}$.

   (3.) $\text{Sin A} \cdot \sin (A + 2C) + \sin B \cdot \sin (B + 2A) + \sin C \cdot \sin (C + 2B) = 0$, if $A + B + C = 180°$.

5. Explain the notation $\sin^{-1}s$, $\tan^{-1}t$.

   Prove—

   (1.) $\text{Sin}^{-1}\dfrac{3}{5} + \sin^{-1}\dfrac{4}{5} = 90°$.

   (2.) $\text{Tan}^{-1}\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} + \tan^{-1}\sqrt{\dfrac{(s-a)(s-c)}{s(s-b)}} + \tan^{-1}\sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}} = 90°$

   when $s = \dfrac{a+b+c}{2}$.

6. Show that $\dfrac{\text{arc}}{\text{radius}}$ may be properly taken for the measure of an angle; of what angle is 1·5708 the circular measure.

   If (θ) be the circular measure of an angle less than a right angle.

   Prove $\sin\theta$ less than $\theta - \dfrac{1}{4}\theta^3$, and verify this when $\theta = \dfrac{\pi}{6}$.

7. In a plane triangle of which A, B, C are the angles, and (a), (b), (c) the sides subtending them.

   Prove—

   (1.) $c = a \cos B + b \cos A$.

   (2.) $\frac{a}{b} = \frac{\sin A}{\sin B}$.

   Deduce from these expressions the value $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

   Prove $a^2 + b^2 + c^2 = 2ab \cos C + 2ac \cos B + 2bc \cos A$.

8. Find the area of a triangle in terms of any two sides and their included angle, and also in terms of the three sides. Find the area of the greatest triangle that has for two of its sides 50 and 60 feet.

9. Given the difference between the angles at the base of a triangle 17° 48′, and the sides subtending those angles 105·25 feet and 76·75 feet: find the angle included by the given sides.

10. An observer in a balloon when it is one mile high observes the angle of depression of a conspicuous object on the horizontal ground to be 35° 20′, then after ascending vertically and uniformly for 20 minutes, he observes the angle of depression of the same object to be 55° 40′: find the rate of ascent of the balloon in miles per hour.

11. A tower which stands on a horizontal plane is 200 feet high, and there is a small loophole in the tower at a certain height above the ground; an observer is at a horizontal distance from the tower of 300 feet, but stands on a mound so that his eye is 12 feet above the ground on which the tower stands, and in that position, the angles subtended at his eye by the portions of the tower above and below the loophole are equal: find the height of the loophole from the ground.

12. In a circle which has a radius of 10 feet two chords AB, CD are drawn at right angles to each other, and intersecting in O, AO and CO are three and four feet respectively: find the sides and angles of the quadrilateral ACBD formed by joining the extremities of the chords.

PURE MATHEMATICS. (1.)

1. If two straight lines be at right angles to the same plane, they shall be parallel to one another.

2. If two straight lines be cut by parallel planes, they shall be cut in the same ratio.

3. If two straight lines do not intersect, and along one of them AB, BC, CD, &c. be taken all equal to one another, and along the other *ab*, *bc*, *cd*, &c. all equal to one another, the straight lines A*a*, B*b*, C*c*, &c. shall be all parallel to a certain plane.

4. Define a logarithm, and prove that

$$\log_a N = \frac{\log_b N}{\log_b a}.$$

Prove that if $\log_a b = m$ and $\log_b a = n$, then

$$\left(\frac{\log_a m}{\log_b n}\right)^2 = \frac{m}{n}.$$

5. Find the expansion of $\log_e (1 + x)$ in ascending powers of x.

Prove that the coefficient of x^n in the expansion of

$$\log (1 + x + x^2) \text{ is } \frac{-2}{n} \text{ or } +\frac{1}{n}$$

according as n is or is not, a multiple of 3.

6. Prove that

$$\cos \alpha + \sqrt{-1} \sin \alpha = e^{\alpha\sqrt{-1}}$$

hence prove that

$$\cos\left(\frac{\pi}{4} + \theta\sqrt{-1}\right) = \frac{1}{2\sqrt{2}}\left\{(e^{\theta} + e^{-\theta}) - \sqrt{-1}\,(e^{\theta} - e^{-\theta})\right\}$$

7. If x be the circular measure of an angle, prove that

$$x = \tan x - \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} - \&c.$$

Prove that

$$3^{\frac{1}{2}} - 3^{-\frac{1}{2}} - \frac{1}{3}\left(3^{\frac{3}{2}} - 3^{-\frac{3}{2}}\right) + \frac{1}{5}\left(3^{\frac{5}{2}} - 3^{-\frac{5}{2}}\right) - \&c. = \frac{\pi}{6}.$$

8. Find an expression for $\cos^n\theta$ in terms of cosines of descending multiples of θ when n is an even integer.

9. Prove that the foot of the perpendicular from the focus on any tangent to a parabola lies on the tangent at the vertex.

A long rectangular slip of paper is folded so that one of the corners always lies on the opposite side. Prove that the crease always touches a parabola of which the aforesaid side is the directrix.

10. In the ellipse prove that if CP and CD be conjugate semi-diameters

$$CP^2 + CD^2 = CA^2 + CB^2.$$

11. In the hyperbola the foot of the perpendicular from the focus on the tangent lies on the circle whose diameter is the transverse axis.

    What portion of this circle corresponds to either branch of the hyperbola?

12. Show how to pass from one set of rectangular axes to another without changing the origin.

    Find the angle through which the axes must be turned in order that the equation

    $$Ax + By + C = 0$$

    may be reduced to the form x = constant, and find the value of this constant. What is the geometrical meaning of this result?

13. Find any form of the equation of the ellipse, and interpret the constants.

    A number of ellipses are described upon the same major axis, prove that the locus of the extremities of their latera recta is a parabola.

14. Find the equation of the tangent at any point of an hyperbola and deduce that of the normal.

    If the hyperbola be equilateral, prove that the portion of the normal intercepted between the axes is bisected by the curve.

PURE MATHEMATICS. (2.)

1. ACB, ADB are two segments of circles on the same base AB; if through any point C in the arc ACB, two straight lines ACD, BCE be drawn to meet the arc ADB in D and E, prove that the arc DE is of constant length.

2. If two circles intersect each other, prove that their common chord bisects their common tangent.

3. Having given the base of a triangle, and the ratio of the sides, prove that the locus of the vertex is a circle.

4. If $\frac{1}{a} + \frac{1}{c} = \frac{1}{b-a} + \frac{1}{b-c}$, prove that a, b, c are in harmonical progression.

5. If $\tan^2\theta - \sec^2 a = 1$, prove that
$$\sec\theta + \tan^3\theta \operatorname{cosec}\theta = (3 + \tan^2 a)^{\frac{3}{2}}.$$

6. If the sides of a triangle be in Arithmetical progression, the cotangents of its semi-angles will also be in Arithmetical progression.

7. A man walking towards a tower on the edge of which a flagstaff is fixed, observes that when his distance from the tower is c feet, the flagstaff subtends the greatest angle at his eye, and he finds that this angle is a; prove that the height of the flagstaff is $2c \tan a$.

8. Find the present value of an annuity of £A per annum, to continue for n years, allowing compound interest.

   What is the present value of a perpetual annuity of £100 a year, to commence at the end of two years from the present time, allowing compound interest at 4 per cent. per annum?

9. Prove that the convergents to a continued fraction, taken in order, are alternately less and greater than the continued fraction.

   Find the continued fraction equivalent to $\sqrt{10}$ and its first four convergents.

   Prove that the difference between any two consecutive odd convergents to $\sqrt{a^2+1}$ is a fraction whose numerator is $2a$.

10. If a, b, and c be positive integers, prove that one solution of the equation $ax - by = c$ can always be found such that x and y are positive integers, and hence find the general solution in positive integers. Prove also that the successive values of x are in an arithmetical progression of which b is the common difference.

    Find the least solution in positive integers of the equation
$$15x - 19y = 10.$$

11. Find all the solutions, in positive integers, of the equation
$$xy - 2x - y = 8.$$

12. Find the sums of the series

$$(\alpha)\ 1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \ldots + n \cdot 1.$$

$$(\beta)\ 1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \frac{5}{2^4} + \ldots \text{ to infinity.}$$

13. Find the number of cannon balls in a pyramidal pile: 1st. When the base is an equilateral triangle having n balls in a side: 2nd. When the base is a square having n balls in a side. Hence show that a square pile is equivalent to two triangular piles.

14. If the fraction $\frac{x^3 - 7x + 13}{x^2 - 7x + 12}$ be expanded in ascending powers of x, find the coefficient of x^3 and of x^r.

15. A sum of money is distributed amongst a certain number of persons. The second receives one shilling more than the first, the third two shillings more than the second, the fourth three shillings more than the third, and so on. If the first person receives one shilling and the last £3. 7*s*., what is the number of persons, and the sum distributed?

16. If an event may happen in different independent ways, prove that the probability of its happening is the sum of the probabilities of its happening in the different ways.

Find the chance of throwing 6 with two dice. Find also the odds against throwing 6 twice at least in three throws with two dice.

PURE MATHEMATICS. (3.)

1. If a be a root of the equation $f(x) = 0$, prove that $f(x)$ is divisible by $x - a$.

   Find the ratio of b to a in order that the equations $ax^2 + bx + a = 0$ and $x^3 - 2x^2 + 2x - 1 = 0$ may have either one or two roots in common.

2. If p be the numerically greatest of the negative coefficients in the equation

$$x^n + p_1 x^{n-1} + \&c. + p_n = 0.$$

   Prove that $p + 1$ is a superior limit of the positive roots.

3. If $f(x) = 0$ have equal roots, prove that the equations $f(x) = 0$ and $f'(x) = 0$ must have a root in common;

   Hence prove that the equation

$$x^n + nx^{n-1} + n \cdot (n-1) \cdot x^{n-2} + \&c. + n \cdot (n-1) \ldots\ldots 2 \cdot 1 = 0$$

   cannot have equal roots.

4. Define a differential coefficient, and from your definition find the differential coefficient of a quotient of one given function of x divided by another.

   Find also, from first principles, the differential coefficients of $\log_e x$ and $\tan x$.

5. Differentiate—

$$(1)\ \log e^x \cdot \left(\frac{x-2}{x+2}\right)^{\frac{3}{4}} \qquad (2)\ \tan^{-1}\frac{1-\cos x}{1+\cos x}$$

   and find $\frac{dy}{dx}$ when $x^{y \log x} = y^{x \log y}$.

6. If $u = \sqrt{\sin^{-1} x}$, prove that

$$\frac{d}{dx}\log\left(u\frac{du}{dx}\right) = \frac{x}{1-x^2}.$$

7. Prove that $f(a+h) - f(a) = hf'(a+\theta h)$, θ being some fraction less than unity, and $f(x)$ and $f'(x)$ being finite between the limits a and $a + h$ of x.

   Assuming Maclaurin's theorem, prove that θ is very nearly equal to $\frac{1}{2}$ when h is very small.

8. If $e^y = e + xy$, find the first six terms of the expansion of y in powers of x.

9. If ABC be a triangle inscribed in a given circle, and the side BC be of constant length, prove that any small variations db, and dc in the lengths b and c of the remaining sides are connected together by the equation $\frac{db}{\cos B} + \frac{dc}{\cos C} = 0$.

10. Find the value of

(1) $\dfrac{x^2 \cos x - \sin^2 x}{\dfrac{x^2}{2} - 2\sin^2 \dfrac{x}{2}}$ when $x = 0$.

(2) $(x^2 + 3x - 1)^{\frac{1}{x}}$ when $x = \infty$.

11. Find the conditions that $f(x)$ may be a maximum or minimum when $x = a$.

If P and Q be two fixed points situated either, both without, or both within, the circumference of a circle whose centre is O, and if R be a point upon that circumference, prove that the sum of the distances PR and QR is the least possible when PR and QR make equal angles with the radius OR.

12. Find expressions—

(1) For the perpendicular from the pole upon the tangent,
(2) For the radius of curvature,

at any point of a curve referred to polar co-ordinates.

If S be the pole, Y the foot of the perpendicular (p) from S on the tangent PY at P, and ϕ the angle between SY and any fixed line, prove that $\dfrac{dp}{d\phi} = \text{PY}$.

13. Find the asympotes to the curve

$$(y - 2x)(y^2 - x^2) = a(y^2 - 9x^2),$$

and the form of the curve at the origin.

14. Integrate—

(1) $\dfrac{dx}{x^2 - 5x + 6}$. (2) $\sqrt{a^2 - x^2}\,.\,dx$.

(3) $\dfrac{dx}{\cos x}$.

15. Find a formula of reduction for the integral

$$\int x^m \sqrt{2ax - x^2}\,dx$$

and find the value of the definite integral

$$\int_0^{2a} x\sqrt{2ax - x^2}\,dx.$$

STATICS.

1. State under what circumstances two forces have a single resultant.

   Two forces, whose magnitudes are P lbs. and $P\sqrt{2}$ lbs., act at a point in directions inclined to each other at an angle of 135°; find the direction and magnitude of their resultant.

2. Define a couple, and the moment of a couple.

   Prove that the effect of a couple is not altered if its arm be turned about one end on the plane of the couple through any angle.

   Find the resultant of any number of couples acting in one plane.

3. Enunciate the triangle of forces.

   P is a particle acted upon by forces emanating from two fixed centres A and B, and proportioned to the distances from those points respectively; if the force from A be repulsive and the force from B attractive, prove that the resultant force on P is constant in magnitude and direction.

4. The three perpendiculars from the angular points of a triangle ABC upon the opposite sides meet in the point T, and O is the centre of the circle circumscribing the triangle; prove that if three forces acting at a point be represented in magnitude and direction by the lines AT, BT, and CT, their resultant is represented in magnitude and direction by twice the line OT.

5. It is required to hang a picture on a vertical wall so that it shall lean forwards at a given angle to the wall, and the picture is to be supported by a single cord attached to a point of the wall at a given height above its lowest edge; determine by a geometrical construction the point on the back of the picture to which the cord must be attached, and find the length of cord which will be wanted.

6. Define a lever, and give instances of different kinds of levers.

   Describe the action of the muscles of the arm when it is held out horizontally from the elbow and a weight is held in the hand.

7. Prove that a system of heavy particles has one and only one centre of gravity. A straight rod, one foot in length, and weighing one ounce, has an ounce of lead fastened to it at one end, and another ounce of lead fastened to it at a distance from that end equal to one-third of its length; find its centre of gravity.

8. Two equal rods AB, BC are jointed at B, and have their middle points connected by an inelastic string of such a length, that when it is straightened, the angle ABC is a right angle; if the system be suspended by a string attached to the point A, prove that the inclination of AB to the vertical will be $\tan^{-1}\frac{1}{3}$, and find the tension of the string and the action at the joint.

9. Define the horse-power of an engine. An engine is required to raise a weight of a ton from a depth of 150 fathoms in five minutes; what is its horse-power?

10. Find the ratio of the Power to the Weight when there is equilibrium on the Wheel and Axle.

    If the radius of the wheel be six times that of the axle, and if by means of a power of 5 lbs. a weight is lifted through 50 feet, find the amount of energy expended.

11. Find the ratio of the Power to the Weight on a smooth Screw, the axis of which is vertical.

    If the power be applied at the end of an arm one foot in length, and if the screw make ten complete turns in one inch of its length, find the power which will support a weight of a ton.

12. Describe a method of determining practically the coefficient of friction between two given substances.

    A heavy body rests on a rough inclined plane; if $\tan \beta$ be the coefficient of friction and α the inclination of the plane, prove that the greatest and least force acting upwards parallel to the plane, which will support the weight are in the ratio of

    $$\sin \overline{\alpha + \beta} \text{ to } \sin \overline{\alpha - \beta}.$$

13. Explain the use of a balancing pole to a man standing on a tight rope.

DYNAMICS.

NOTE.—*In all the following questions the measure of the force of gravity, when required, may be taken as* 32 *feet.*

1. When a body's motion is variable, how is the velocity at any point of its course estimated? What will be the law of the motion of a particle acted on by a single uniform force in the direction of its motion? A particle under the action of a force in its direction has a velocity of 200 feet at the end of the third second, and of 260 feet at the end of the fourth second, what will be its velocity at the end of the fifth second if the force be uniform? Account briefly for the variation in the measure of the force of gravity at different stations on the earth.

2. If a particle be projected in the direction of a uniform force (f), with a velocity V, investigate the following expressions for the space (s) described, and the velocity (v) acquired, after (t) seconds.

   (1) $s = Vt + \frac{ft^2}{2}$. (2) $v + V = ft$, hence find (v) in terms of s.

   What will (f) be in case of a body sliding down a smooth inclined plane under the action of gravity?

3. Divide a given inclined plane into three parts, so that a particle at the top of the plane sliding from rest may describe the three parts in equal times. In what number of seconds will each part be described if the height of the plane is 64 feet, and its length 288 feet?

4. Enunciate the second law of motion, and apply it to determine the position of a body, acted on by gravity, projected with a given velocity at a given angle of projection after it has been in motion for t seconds.

   If (α) be the angle and V the velocity of projection, find the position after a time represented by $\frac{2V \sin \alpha}{g}$.

5. Find the equation to the path of a projectile in vacuo, when referred to horizontal and vertical co-ordinates.

   From the equation obtained, determine the range on an inclined plane that passes through the point of projection, and show that the vertical line, drawn from the point in the curve at which the projectile is moving parallel to the inclined plane, will pass through the middle point of the range on that plane.

6. The third law of motion is sometimes enunciated with reference to "action" and "re-action," and sometimes with reference to "momentum"; give both enunciations, and show how they become consistent with each other.

   Find the pressure which, acting horizontally for 5 seconds on a weight of 160 pounds, placed on a smooth table, will generate a velocity of 15 feet per second.

7. When a weight (P) lifts another weight (W) by means of a string passing over a fixed pulley, find the accelerating force, and the tension of the string.

   If P be 45 pounds and W 36 pounds, what weight suspended from a string at rest will produce the same tension as is produced by P lifting W?

8. How is an impulsive force estimated? Give some account of the action of one smooth elastic ball on another during direct impact. State a general principle from which the motion of two balls after impact may be obtained.

   A given smooth elastic ball impinges directly with a given velocity upon another at rest, find the velocities of each ball after impact, and show that the velocity of the centre of gravity of the two balls before and after impact is the same.

9. Explain generally what is meant by centrifugal force. If a weight at the end of a string be whirled round a fixed point in the string with a given angular velocity, show how to find the tension of the string.

   A string $2\frac{1}{2}$ feet long can just support five pounds without breaking; if a weight of *one* pound be attached to the end of the string, which is whirled round its other end uniformly in a horizontal plane, how many revolutions per minute must it make so as just to break the string?

10. What is a simple pendulum? Find the time of the oscillation of a simple pendulum in a small circular arc, and determine approximately the length of the seconds pendulum.

    If a seconds pendulum be lengthened one inch, find the number of seconds it will lose in 12 hours.

SET II.

EUCLID.

1. Construct a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third.

   How would the process fail if the last condition were not fulfilled?

2. Parallelograms on the same base and between the same parallels are equal to each other.

   Show that if two triangles have two sides of the one equal to two sides of the other each to each, and the sum of the two included angles equal to two right angles, the triangles are equal.

3. In a right-angled triangle the square on the side subtending the right angle is equal to the sum of the squares on the sides containing the right angle.

   Show how to construct a straight line, the square on which shall be any given multiple of a given square.

4. If a straight line be bisected and produced to any point the rectangle contained by the whole line thus produced, and the part produced, together with the square on half the line bisected is equal to the square on the straight line which is made up of the half and the part produced.

5. The angle at the centre of a circle is double of the angle at the circumference on the same base that is on the same arc.

6. If two straight lines cut one another within a circle the rectangle contained by the segments of the one shall be equal to the rectangle contained by the segments of the other.

   In a given straight line AB, find a point O such that the rectangle contained by the segments AO and OB shall be equal to a given rectangle not greater than the square on half of AB.

7. Describe a circle about a given triangle.

   Lines are drawn parallel to one side of a triangle cutting the other sides or the other sides produced. Show that if circles be described about the new triangles so formed, their centres all lie in a straight line, and find the angles which this line makes with the two sides of the original triangle.

8. If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the segments of the base shall have the same ratio which the other sides of the triangle have to one another.

   If ABC be the triangle, A the vertical angle, BE and CE the segments of the base, and if circles be described about the triangles ABE, ACE, the diameters of these circles are to each other in the same ratio as the segments of the base.

9. If four straight lines be proportionals, the rectangle contained by the extremes shall be equal to the rectangle contained by the means.

   On a given straight line describe an isosceles triangle equal to a given triangle.

10. Similar triangles are to each other in the duplicate ratio of their homologous sides.

11. Given a point O in the line AB, find two other points CD, such that a line OP given in direction shall bisect the angle CPD, and the segments CO, OD shall bear a given ratio to each other.

ALGEBRA.

1. "The product of two algebraical quantities having like signs is positive, and having unlike signs negative." State briefly the steps by which this rule is obtained, and show its arithmetical correctness when in the product obtained for $(a - b) \times (c - d)$, $a = 12$, $b = 9$, $c = 10$, $d = 4$.

   If x and y differ in value, explain why $(x - y)^2 = (y - x)^2$. Show how the rule of signs stated above leads to the introduction of impossible or imaginary quantities in algebra.

2. Multiply—

   $(1 + x + 2x^2)^2 - (1 - x - 2x^2)^2$ by $(1 + x - 2x^2)^2 - (1 - x + 2x^2)^2$.

   Find the product of

   $$x^2 - (a + b)x + ab \text{ by } x^2 + (a - b)x - ab,$$

   and examine what the product becomes if in it either (a) or (b) be substituted for x.

3. Divide $x^4 - 2bx^3 - (a^2 - b^2)x^2 + 2a^2bx - a^2b^2$ by

   $$x^2 - (a + b)x + ab.$$

   Reduce to its simplest form—

   $$\frac{(x - y)(y - z) + (x - y)(z - x) + (y - z)(z - x)}{x(z - x) + y(x - y) + z(y - z)}.$$

4. If (m) be a whole number, prove that $(x + y)$ is divisible by $x^{\frac{1}{m}} + y^{\frac{1}{m}}$, when (m) is odd.

   Write down the three last terms of the quotient of $(x + y)$ divided by $(x^{\frac{1}{2m+1}} + y^{\frac{1}{2m+1}})$, and examine how many terms the quotient will contain.

5. Find the fraction in its lowest terms which is the square root of

   $$\frac{1 + 4x^{\frac{1}{2}} - 2x - 12x^{\frac{3}{2}} + 9x^2}{1 - 4x^{\frac{1}{2}} + 6x - 4x^{\frac{3}{2}} + x^2}.$$

6. Solve the following equations:—

   (1.) $(x - 4)(x - 8)(x - 10) = (x - 6)(x - 7)(x - 9)$.

   (2.) $5x^2 - 14x = 3$.

   (3.) $2cx^2 - abx + 2abd = 4cdx$.

   (4.)
   $$\left.\begin{aligned} (x + y)^2 - z^2 &= 65 \\ x^2 - (y + z)^2 &= 13 \\ x + z - y &= 9 \end{aligned}\right\}.$$

7. Form a quadratic equation whose roots are $-5 \pm 6\sqrt{-1}$.

   If $x^2 - px + q = 0$ and $x^2 - p_1x + q_1 = 0$ have a common root, determine it, and in that case show that

   $$(q - q_1)^2 = (p - p_1)(p_1q - pq_1).$$

8. A bill of £63. 5*s.* was paid in sovereigns and half crowns, and the number of coins used in the payment was 100; how many sovereigns were paid, and how many half crowns?

9. A cask (A) is filled with 50 gallons of water and a cask (B) with 40 gallons of brandy; (C) gallons are drawn from each cask, mixed, and replaced. The same operation is repeated. Find (C) when there are $8\frac{1}{9}$ gallons of brandy in (A) after the second replacement.

10. Find the sum of (n) terms of the series $(a - b) + (a - 3b) + (a - 5b)$ &c., without assuming a formula for the sum of an arithmetic series, and apply the result to find the sum of 10 terms of $76 + 70 + 64 +$ &c.

11. The sum of the two first terms of a geometric series is 12 and of the three first terms is 39, find the series, and determine if the condition is satisfied by more than one series.

    Find the sum of the infinite series

    $$1 + r + (1 + b)r^2 + (1 + b + b^2)r^3 + \text{\&c.},$$

    (r) and (b) being proper fractions.

12. Show how to transfer a whole number from one scale of notation to another, and prove that if (r) be the radix of the scale, the number itself will be divisible by $(r + 1)$ if the difference between the sum of the digits in the odd places and the sum of the digits in the even places be divisible by $(r + 1)$.

13. Find the number of different permutations of (n) letters taken altogether when one letter occurs (p) times and another (q) times in each permutation.

    If the n letters contain only a and b, and n be even, show that the number of permutations will be greatest when $p = q = \frac{n}{2}$.

14. Assuming the form of the binomial theorem, find the greatest term of $(1 + x)^n$ when (n) is a positive integer and x is also positive. Apply the result to the determination of the greatest term of $\left(1 + \frac{5}{2}\right)^6$, and express the value of the greatest term.

    Show by actual multiplication that the whole coefficient of x^4 in the product of the expansions of $(1 + x)^n$ and $(1 - x^n)$ is equal to the coefficient of x^4 in the expansion of $(1 - x^2)^n$.

TRIGONOMETRY.

1. Define the degree and the unit of circular measure, and show how to pass from one of these units to the other.

   If the radius of a circle be 25 feet, find to four decimal places the length of the arc subtending an angle of 3°, taking π to be 3·1416.

2. Define the principal trigonometrical ratios, and trace the changes in sign and magnitude of $\frac{\sin 3\theta}{\cos 2\theta}$ as θ varies from 0 to $\frac{\pi}{2}$.

3. Prove that $\sin(A + B) = \sin A \cos B + \cos A \sin B$ where A and B are positive angles each less than 90°.

   Assuming that the formula holds for all values of the angles deduce that for $\cos(A - B)$.

4. Find cos 3A and tan 3A in terms of cos A and tan A respectively.

5. Prove that for all values of A—

   (1.) $2 \operatorname{cosec} 4A + 2 \cot 4A = \cot A - \tan A$;

   (2.) $\frac{\cos A + \cos 3A}{\cos 3A + \cos 5A} = \frac{1}{2 \cos 2A - \sec 2A}$.

6. Prove that

$$\tan^{-1}m + \tan^{-1}n = \cos^{-1}\frac{1 - mn}{\sqrt{(1 + m^2)(1 + n^2)}}$$

7. Find sin 18°.

   If $\cos A = \tan B$ and $\cos B = \tan A$;
   Prove that $\cos A = 2 \sin 18°$.

8. Prove the formula

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

   and apply it to prove that if the straight line which bisects the vertical angle of a triangle also bisects the base, then the triangle must be isosceles.

9. Find the area of a triangle in terms of the sides.

10. Find the radius of the circle which touches one side of a triangle and the two other sides produced.

    If ABC and DEF be two triangles with the same perimeter, prove that the radius of the escribed circle touching the side BC of the first is to the radius of the escribed circle touching the side EF of the second as $\tan\frac{A}{2}$ is to $\tan\frac{D}{2}$.

11. Given two sides b and c, and the included angle A of a triangle, find a formula for determining the side a, without the previous determination of the angles B and C, adapted to logarithmic computation.

    Apply to the case of b = 354 yds., c = 426 yds., A = 49° 16′.

12. A building on a square base ABCD has two of its sides AB and CD parallel to the bank of a river.

An observer standing on the river's bank in the same straight line with DA finds that the side AB subtends, at his eye, an angle of 45°.

Having walked (a) yards along the bank he finds that the side DA subtends an angle whose sine is $\frac{1}{3}$.

Prove that the length of each side of the building is $\frac{a}{\sqrt{2}}$ yards.

PURE MATHEMATICS (1).

1. If two parallel planes be cut by another plane, their common sections with it are parallel.

   If a straight line CD be parallel to a plane, prove that the intersection of any plane passing through CD with the given plane will be a straight line parallel to CD.

2. Define a solid angle. When are solids said to be similar?

   If a solid angle be contained by any number of plane angles, these together shall be less than four right angles.

   Is this proposition true if any of the planes which contain the solid angle make re-entering angles with one another?

3. Define a prism and a pyramid, and show that the volume of a pyramid is a third part of a prism which has the same base and altitude.

   The base of a right pyramid is a regular hexagon inscribed in a circle whose radius is 10 inches; find the ratio of the contents of the pyramid and of a cone having the circle for its base, the altitudes both of the cone and pyramid being 12 inches.

4. In any plane triangle of which A, B, C are the angles, and (a), (b), (c) the sides subtending them:

   Prove—

   (1.) $c^2 = (a+b)^2 \sin^2\frac{C}{2} + (a-b)^2 \cos^2\frac{C}{2}$.

   (2.) $\sqrt{bc \sin B \sin C} = \frac{b^2 \sin C + c^2 \sin B}{b+c}$.

5. If $x + \frac{1}{x} = 2\cos\theta$, and $y + \frac{1}{y} = 2\cos\phi$,

   prove (1.) $x^m + \frac{1}{x^m} = 2\cos m\theta$,

   (2.) $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$.

6. Investigate the expression

   $$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right) \&c.$$

   and hence prove $\pi = 2 \cdot \frac{2^2}{1.3} \cdot \frac{4^2}{3.5} \cdot \frac{6^2}{5.7} \cdot \frac{8^2}{7.9}$, &c.

7. Find the sum of *either* of the two following series:—

   (1.) $\sin\theta + \sin 2\theta + \sin 3\theta + \&c. + \sin n\theta$.

   (2.) $\cos\theta + \cos 2\theta + \cos 3\theta + \&c. + \cos n\theta$.

8. If A be the vertex of a parabola, S the focus, P any point on the curve, T the point where the tangent at P meets the axis produced, PQ perpendicular to the directrix:
   Prove—
   (1.) PT bisects the angle SPQ.
   (2.) The quadrilateral SPQT is a rhombus.

9. Prove that the perpendiculars drawn from the foci of an ellipse on a tangent at any point intersect the tangent in the circumference of the circle described on the major axis.
   If a tangent be drawn at the extremity of a latus rectum of the ellipse, find the lengths of the perpendiculars upon this tangent from the foci.

10. If a right cone be cut by a plane which meets the cone on both sides the vertex, the section is an hyperbola.

11. Given the equations to two straight lines referred to rectangular co-ordinates, find the angle at which they intersect.
   Prove that the equation
   $$3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0$$
   represents two straight lines at right angles to each other.

12. Prove that the equation $5y^2 + 5x^2 + 2xy - 12x - 12y = 0$ represents an ellipse; find its axes and determine its area.

13. When is an hyperbola said to be equilateral or rectangular? Find the eccentricity of such an hyperbola. In a rectangular hyperbola, the distance of any point of the curve from the centre is a mean proportional between the distances of the same point from the foci.

PURE MATHEMATICS (2).

1. Prove that the sum of the squares on the three straight lines drawn from the angles of a triangle to the points of bisection of the opposite sides is equal to three-fourths of the sum of the squares of the sides of the triangle.

2. Given one side of a triangle, the angle opposite to that side and the point at which the inscribed circle touches that side; construct the triangle.

3. Semicircles BDB and CEA are described upon the sides BA and AC of the right-angled triangle BAC, and another semicircle BFAGC is described upon the hypotenuse BC.
   Prove that the sum of the areas of the two curvilinear figures BDAF and CEAG is equal to the area of the triangle ABC.

4. If $a = xy^{p-1}$ $\quad b = xy^{q-1}$ $\quad c = xy^{r-1}$,
   Prove that $a^{q-r}b^{r-p}c^{p-q} = 1$;
   and if $\quad 2s = a + b + c$, prove that
   $(s-a)^3 + (s-b)^3 + (s-c)^3 + 3abc = s^3$.

5. If x be a possible quantity, prove that
$$\frac{2x^2 + 6x + 3}{2x + 1}$$
must be either less than 1 or greater than 3.

6. If the m^{th} term of a harmonic progression be equal to n, and the n^{th} term be equal to m, prove that the $(m+n)^{\text{th}}$ term is equal to $\frac{mn}{m+n}$.

7. Expand $\frac{1}{(1-x)^2}$ in ascending powers of x, and prove that if x be equal to $\frac{n+1}{n}$ the expression will be equal to the sum of the first n terms of the expansion.

8. If n be an even number, prove that the sum of the series
$$1 + 2^2 + 3 + 4^2 + \&c.$$
to n terms is
$$\frac{n(n+4)(2n+1)}{12},$$
and find the sum of n terms of the series
$$\frac{1}{1\cdot 6} + \frac{1}{6\cdot 11} + \frac{1}{11\cdot 16} + \&c.$$

9. Find all the solutions in positive integers of the equation
$$xy + x + y = 35.$$

10. Express $\sqrt{14}$ in the form of a continued fraction, and find the values of

$$\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2}}}$$

2, &c., ad infinitum.

11. Three men toss a guinea in succession on the condition that he who first tosses heads keeps the coin.

Prove that the value of the expectation of the man who begins to toss is 12*s*.

12. A debt is to be paid by equal annual instalments extending over 20 years.

If the rate of interest be 5 per cent., prove that each annual payment is just a very little more than 8 per cent. of the debt.

13. If α and β be two values of θ satisfying the equation

$$\frac{\cos\theta}{a} + \frac{\sin\theta}{b} = \frac{1}{c},$$

prove that $a\cos\frac{\alpha+\beta}{2} = b\sin\frac{\alpha+\beta}{2} = c\cos\frac{\alpha-\beta}{2}$.

14. Equilateral triangles DBC and EBC are described upon opposite sides of BC one of the sides of the triangle ABC.

Prove that $AD^2 + AE^2 = AB^2 + AC^2 + BC^2$.

15. If in the triangle ABC the angle A be three times the angle B, prove that

$$\sin B = \frac{1}{2}\sqrt{\frac{3b-a}{b}}.$$

16. Prove that

$$\log(x + y\sqrt{-1}) = \frac{1}{2}\log(x^2+y^2) + \tan^{-1}\frac{y}{x}\sqrt{-1}.$$

PURE MATHEMATICS (3).

1. Prove that an equation of even degree which has the last term of contrary sign to the first has at least two real roots of contrary signs.

2. Transform the equation $x^3 - \frac{x}{4} - \frac{3}{4} = 0$ into one whose roots exceed by $\frac{3}{2}$ the corresponding roots of the given equation.

3. Prove that in any equation, complete or incomplete, the number of positive roots cannot exceed the number of changes in the signs of the coefficients.

4. It being given that the limit of $(1+x)^{\frac{1}{x}}$ when x is indefinitely increased is ϵ, show that the limit of $(1+ax)^{\frac{1}{x}}$ is ϵ^a.

5. Show that the differential coefficient with respect to x of the product of u and v, two functions of x, is $u\frac{dv}{dx} + v\frac{du}{dx}$.

   Differentiate the following functions with respect to x, viz.:

   $$\tan^{-1}\frac{a^2}{x^2} \text{ and } \sqrt{\frac{1+\sin x}{1-\sin x}} + \sqrt{\frac{1-\sin x}{1+\sin x}}.$$

6. If u be any function of $(at + x)$ prove that $\frac{d^n u}{dt^n} = a^n \frac{d^n u}{dx_n}$.

7. Prove that if $f(x)$ has a maximum or minimum value when $x = a$ then $f'(a) = 0$.

   Find a value of x for which $x^m(1-x)^n$ is a maximum, m and n being positive.

8. Evaluate the following indeterminate forms:—

   $$\frac{2}{x^2-1} - \frac{1}{x-1} \text{ when } x = 1,$$

   $$\frac{\epsilon^x - \epsilon^{-x}}{\sin x} \text{ when } x = 0.$$

   If ABCD be a quadrilateral figure inscribed in a circle whose diagonals intersect in O, and if, while A, B, and D remain fixed, C approaches B, find the limiting value of $\frac{CO}{BO}$.

9. Find the equation to the tangent through a given point to the curve $y = f(x)$.

   The equations to the tangents drawn from the origin to the parabola $ay = x^2 + bx + c^2$ are $y = \frac{b \pm 2c}{a} x$.

10. If the logarithmic curve $y = a\epsilon^{\frac{x}{c}}$ has contact of the second order with the curve $\phi(x, y) = 0$, show that the subtangent of the latter curve has a maximum or minimum value at the point of contact.

11. Find the radius of curvature at any point of the polar curve $r = a\epsilon^{m\theta}$, and show that it makes a constant angle with the radius vector.

12. Prove the rule for integration by parts, and integrate the following expressions:—

$$\log x\, dx, \qquad \frac{x^3 dx}{\sqrt{a^2 - x^2}}, \qquad \cos^4 x\, dx.$$

13. If $f(x)$, $\phi(x)$, be rational functions, $f(x)$ being of higher dimensions than $\phi(x)$, and if the roots of the equation $f(x) = 0$ are all real and unequal, show how to integrate the expression $\frac{\phi(x)}{f(x)}\, dx$.

Integrate $\frac{9x^2 + 12x + 6}{2x^3 + 5x^2 - x - 6}\, dx$.

14. Find the length of the arc of a parabola from the vertex to the ordinate y.

DYNAMICS.

1. Define the terms force, acceleration, unit of acceleration.

   If the unit of velocity be that with which a point passes over a feet in t seconds, and the unit of acceleration that which produces in t seconds a velocity of b feet per t seconds, find the units of space and time.

   What space will a particle, moving under a constant acceleration of 32·2 feet per second, describe in the third minute of its motion from rest?

2. Establish the relations between space and time, time and velocity, velocity and space, in the case of a heavy body falling freely in vacuo.

   If s_1 be the space described in t seconds by a body projected directly towards a centre of uniform attractive force; s_2 the space which, if the force then ceased to act, the body would describe in the next t seconds; and s_3 the space which it would describe from rest in t seconds; prove that $s_3 = s_1 + s_2$.

3. Show that if three velocities, represented in direction and intensity by the three sides of a triangle taken in order, be communicated simultaneously to a point, the point will remain at rest.

   Give a practical illustration.

4. Find the velocity acquired, and the time occupied in sliding down a rough inclined plane, the coefficient of friction during motion being constant.

   Show that the points in a vertical plane, from which a particle will slide down a rough chord to a fixed point in a given time, lie in one or the other of two fixed circles.

5. Two particles, of mass p and q respectively, are connected by a string passing over a fixed pulley; determine the motion of the system, and find the tension of the string for a given velocity.

   Show that the acceleration of the common centre of gravity of the particles is $g\left(\frac{p-q}{p+q}\right)^2$.

6. Prove that particles projected obliquely in vacuo from a given point, and moving in the same vertical plane, describe parabolas having a common directrix.

   If v_1, v_2 are the velocities at the extremities of any arc of the path of a projectile, α the angle between the tangents at those points, and v the velocity at the vertex of the parabola; prove that the time of describing the arc $= \frac{v_1 v_2 \sin \alpha}{gv}$.

7. State Newton's Third Law of Motion. How is it exemplified (1) in the bursting of a shell, (2) in the motion of two bodies acted upon by their mutual attractions only?

8. A perfectly elastic ball being placed on a horizontal plane between two vertical hard planes, find by geometrical construction the direction in which it must be propelled in order to strike another ball in a given position after one, two, and three reflexions respectively from the vertical planes.

   If n balls of common elasticity be placed in a row, and the first be made to impinge on the second; and if each ball up to the $(n-1)^{th}$ is brought to rest after impact, compare the masses of the balls, and find the velocity of the n^{th} after impact.

9. Given the accelerations of a particle in two directions at right angles to one another, find the accelerations along the tangent, and normal to its path.

   If a particle attached to a string is made to revolve in a vertical circle, and if the tension of the string at the highest point be zero, what will the tension be at the lowest point?

10. Prove that the time in which a heavy particle will slide down any arc of an inverted cycloid to the lowest point is the same.

    If l_1 = the length of an imperfectly adjusted seconds' pendulum which gains n seconds in an hour, and l_2 = the length of one which loses n seconds at the same place, find the true length of the seconds' pendulum at that place.

11. If a pendulum which beats seconds at the surface of the earth, supposed spherical with a radius of 4,000 miles, loses 10 seconds in 24 hours at the top of a mountain; find the height of the mountain, and the number of seconds which the same pendulum would lose in 24 hours half-way up.

STATICS.

1. By what assumptions does Geometry become applicable to establish the fundamental propositions of Mechanics? *Pressure, tension, reaction, friction* are included under the term force: give examples of the mode of action of each, and state how they are measured statically.

   A weight, at rest, is placed on a smooth table; what forces act upon it? If the table be inclined, show that some other force must be introduced to keep it at rest. If three equal forces act on a particle, how must they be arranged so as to keep it at rest?

2. State the proposition known as the parallelogram of forces, and show how it may be proved by experiment.

   A stream is 96 feet wide; a boat is dragged down the middle of the stream by two men on the opposite banks pulling by ropes each with a pressure of 100 lbs.; the ropes are attached to the same point in the boat, and the length of each rope from the boat to the bank is 60 feet; find the resultant pressure on the boat down the stream.

3. State and prove the condition of equilibrium of a straight lever without weight acted on by any two forces at its extremities.

   AB, a uniform rod of weight (W), movable in a vertical plane about a hinge at A, is sustained in equilibrium by a weight (P) attached to a string BCP passing over a smooth peg C, AC being vertical; if $\cos ACB = \frac{P}{W}$, show that AC = AB.

4. Define the centre of gravity of a solid or system of particles. Show that every solid or system of particles must have a centre of gravity and cannot have more than one such centre.

   Assuming the position of the centre of gravity of a pyramid with a triangular base, show that its position is not altered by placing equal weights in each of the four solid angles of the pyramid.

5. How is the work done by an agent estimated? What is the numerical measure of a horse-power? When weights are raised through different heights, prove that the whole work expended is equal to the work that would be expended in raising a weight equal to their sum through the same height as that through which the centre of gravity of the weights has been raised.

   In how many hours would an engine of 18 horse-power empty a vertical shaft full of water, if the diameter of the shaft be 9 feet, and its depth 420 feet, the weight of a cubic foot of water being $62\frac{1}{2}$ pounds?

6. What is meant by the mechanical advantage of a machine? Is any mechanical advantage obtained in raising a weight by means of a fixed pulley? Investigate the mechanical advantage when a weight is raised by means of a smooth movable pulley,

the parts of the string on each side of the pulley *not* being parallel. At what angle of inclination will the mechanical advantage cease? If the directions of the string on the movable pulley are at right angles to each other, what is the greatest weight that can be raised by a man standing on the ground, whose own weight is 12 stone, when he pulls at the free end of the string which passes over a fixed pulley having its centre on the same horizontal line as the point to which the fixed end of the string is attached?

7. Explain what is meant by the virtual velocity and virtual moment of a pressure acting on a point, and state the principle of virtual velocities.

   If (X) and (Y) be the rectangular components of a pressure (P), show that the virtual moment of P is equal to the sum of the virtual moments of X and Y. When will this equality of virtual moments express the principle of virtual velocities?

8. Define a couple, and show that any system of forces acting in one plane on a rigid body may be reduced to a single force and a couple; hence examine the conditions of equilibrium of the forces.

9. A uniform beam of given weight rests upon a prop with one end against a smooth vertical wall, and a weight is suspended from the other end of the beam; find the position of equilibrium.

10. When friction exists between two substances, explain what is meant by the limited angle of resistance. If the limiting angle of resistance be 30°, what is the coefficient of friction?

    Describe the construction of the screw considered as a mechanical power, and find *either* of the limiting conditions of equilibrium on the screw when friction is taken into account.

11. When is equilibrium said to be stable, unstable, and neutral? Give examples of each kind of equilibrium. If a body rest, with a convex spherical surface on the interior of a concave spherical surface, examine the mathematical condition of stability.

    A sphere of radius (r) rests on a concave sphere, radius (R); if the sphere be loaded so that the height of its centre of gravity from the point of contact is $\frac{4}{3}r$, find (R), in order that the equilibrium may be neutral.

12. State Guldinus's properties, and prove the property for finding *either* the surface *or* the volume of a solid formed by the revolution of a plane curve *or* curvilinear area.

    Determine *either* the volume *or* the surface of a ring made by the revolution of a regular hexagon: the radius of the circle that would circumscribe the hexagon is (a), and (b) is the distance of the centre of the hexagon from the axis about which it revolves.

SET III.

EUCLID.

1. Prove that if from the ends of a side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

   Is it necessary for the truth of this proposition that both lines shall be drawn from the extremities of the base? Can you show any case in which the sum of the two lines drawn from two points in the base to a point within the triangle, exceeds the sum of the two sides of the triangle?

2. Define a parallelogram, and prove that the opposite sides and angles are equal to one another.

   If the opposite sides of a quadrilateral figure be equal, prove that it is a parallelogram.

3. If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it; the angle contained by these two sides is a right angle.

   Find a square equal to the difference of the squares of two given unequal straight lines.

4. Divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts, shall be equal to the square on the other part.

   If the line AB be thus divided at H, and HL be taken in AH, equal to HB, show that AH is similarly divided at L.

5. The angles in the same segment of a circle are equal to one another.

6. If from a point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line, which cuts the circle, and the part of it without the circle, be equal to the square on the line which meets it, the line which meets shall touch the circle.

   Draw two tangents to a circle, which shall contain an angle equal to a given rectilineal angle.

7. Define a locus, and write down the following loci:—
   (1.) Of the vertices of equal triangles upon the same base.
   (2.) Of the vertices of triangles on the same base and with the same vertical angle.
   (3.) Of the centres of all the circles which touch a straight line in a given point.
   (4.) Of the middle points of all equal straight lines in a circle.

8. Describe an isosceles triangle, having each of the angles at the base double of the third angle.

   If the radius of a circle be cut in extreme and mean ratio, the greater segment will be equal to the side of an equilateral and equiangular decagon inscribed in that circle.

9. Give Euclid's definitions of ratio, duplicate ratio, and ex æquali. When is the first of the four magnitudes said to have to the second the same ratio that the third has to the fourth?

10. If the outward angle of a triangle, made by producing one of its sides, be divided into two equal angles by a straight line, which also cuts the base produced; the segments between the dividing line and the extremities of the base have the same ratio which the other sides of the triangle have to one another.

11. When are three lines said to be in harmonical proportion? If both the interior and exterior angles at the vertex of a triangle be bisected by lines which meet the base and the base produced in D, G the segments BG, GD, GC of the base shall be in harmonical proportion.

12. Equal parallelograms which have one angle of the one equal to one angle of the other have their sides about the equal angles reciprocally proportional.

ALGEBRA.

1. Prove that

$$x(y+2)+\frac{x}{y}+\frac{y}{x} \text{ is equal to } a,$$

$$\text{if } x=\frac{y}{y+1} \text{ and } y=\frac{a-2}{2}.$$

2. Multiply together

$$\{(x^2+x+1)a-(x+1)\} \text{ and } \{(x-1)a^2-(x-1)a+3\};$$

and divide

$$(x^3-3x^2y)^2-(3xy^2-y^3)^2 \text{ by } (x-y)^3.$$

3. Find the square root of

$$3(3a^2-2ab+b^2)(a^2+3b^2)+b^2(a+4b)^2.$$

Add a term to ax^2+bx which shall make it a perfect square for all values of x.

4. If x^3+px+r and $3x^2+p$ have a common measure, prove that $\frac{p^3}{27}+\frac{r^2}{4}=0$ and find the common measure in that case.

5. Simplify the expressions

(1.) $$\left(\frac{x^2}{y}+\frac{y^2}{x}\right)\cdot\frac{1}{y^2-x^2}-\frac{y}{x^2+xy}+\frac{x}{xy-y^2}.$$

(2.) $$x^{\frac{1}{2}}y^{\frac{1}{3}}\left(\frac{y^{\frac{1}{4}}}{x^{\frac{1}{3}}}\right)^2\div\frac{y^{-\frac{1}{4}}}{x^{\frac{1}{4}}},$$

and prove that

$$\frac{\sqrt{5+2\sqrt{6}}+\sqrt{5-2\sqrt{6}}}{\sqrt{5+2\sqrt{6}}-\sqrt{5-2\sqrt{6}}}=\frac{1}{2}\sqrt{6}.$$

6. Prove that every quadratic equation has two roots and no more.

If α and β be the roots of

$$ax^2+bx+c=0,$$

prove that

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{b}{c}=0.$$

7. Solve the equations

(1.) $$\frac{3x-2}{4}+\frac{x}{2}-11\frac{5}{6}=\frac{x-\frac{4x-9}{3}}{6}-5.$$

(2.) $$(x^2-bx+b^2)(ax+ab+b^2)=ax^3+a^2b^2+b^4.$$

8. If $2(x^2+a^2-ax)(y^2+b^2-by)=x^2y^2+a^2b^2$, prove that

$$(x-a)^2(y-b)^2+(bx-ay)^2=0,$$

and therefore that $x=a$ and $y=b$ are the only possible solutions.

9. If a, b, c, and d be all positive, prove that $\frac{(a+b)cd}{ad+bc}$ cannot be greater than $\frac{ac+bd}{a+b}$.

10. A man walking from a town, A, to another, B, at the rate of 4 miles an hour, starts one hour before a coach travelling 12 miles an hour, and is picked up by the coach. On arriving at B, he finds that his coach journey has lasted 2 hours. Find the distance between A and B.

11. Find the sum of the series a, $a+b$, $a+2b$, &c. to n terms.

If between every two terms of the above series n arithmetic means be inserted, prove that a new arithmetic series will be formed, in which the number of terms and the sum will be respectively equal to n times the number of terms and sum in the original series.

12. Prove that the fraction $\frac{m_1a_1+m_2a_2 \ldots +m_na_n}{m_1b_1+m_2b_2 \ldots +m_nb_n}$ will be equal to each of the fractions $\frac{a_1}{b_1}$, $\frac{a_2}{b_2}$, &c., if these be all equal to one another, and that it will be intermediate in value between the greatest and least of these fractions if they be not all equal.

13. Find the number of combinations of n things r together, and prove that the total number of combinations of n things 1 at a time, 2 at a time, &c., n at a time, is 2^n-1.

If the $n+1$ numbers a, b, c, d, ... &c. be all different, and each of them a prime number, prove that the number of different factors of the expression a^mbcd ... &c. will be $(m+1)2^n-1$.

14. Prove the binomial theorem for a positive and integral index.

Prove that

$$\left(x+\frac{1}{x}\right)^6-\left(x-\frac{1}{x}\right)^6=12\left(x^4+\frac{1}{x^4}+\frac{10}{3}\right)$$

TRIGONOMETRY.

1. Define the secant, the cosecant, the tangent, and the cotangent of an angle, and prove the relations

$$\text{Sec}^2\text{A} = 1 + \tan^2\text{A}, \quad \text{cosec}^2\text{A} = 1 + \cot^2\text{A}.$$

Find the values of sec 45° and of tan 45°.

2. Trace the changes in the sign and value of the cosine of an angle as the angle changes from 0° to 360°.

Find an expression for all the angles which have the same cosine as a given angle.

What is the general value of θ, 1st, when $\cos\theta = \frac{1}{\sqrt{2}}$,

2nd, when $\cos 4\theta = \frac{1}{\sqrt{2}}$?

3. Having given $\frac{\sin\text{A}}{\sin\text{B}} = \sqrt{\frac{2}{3}}$ and $\frac{\tan\text{A}}{\tan\text{B}} = \frac{1}{\sqrt{3}}$,

find the values of sin A and sin B; and if A and B be two of the angles of a triangle ABC, find the three angles.

4. Taking the case in which A is greater than B and less than a right angle, prove that

$$\cos(\text{A} - \text{B}) = \cos\text{A}\cos\text{B} + \sin\text{A}\sin\text{B}.$$

Having given $\cos\text{A} = \frac{3}{5}$ and $\cos\text{B} = \frac{4}{5}$, find the value of $\cos\frac{1}{2}(\text{A} - \text{B})$.

Prove that if x and y be each less than unity,

$$\sin^{-1}x - \sin^{-1}y = \cos^{-1}\{\sqrt{1 - x^2 - y^2 + x^2y^2} + xy\}.$$

5. Prove the formulæ

(1.) $\frac{\sin\text{A} + \sin 2\text{A}}{\cos\text{A} + \cos 2\text{A}} = \tan\frac{3\text{A}}{2}$.

(2.) $\sin 3\text{A} = 3\sin\text{A} - 4\sin^3\text{A}$.

(3.) $\sin(\text{A} + 45°) + \sin(\text{A} + 135°) = \sqrt{2}\cos\text{A}$.

6. If $\text{A} + \text{B} + \text{C} = 180°$, prove that

(1.) $\sin 2\text{A} + \sin 2\text{B} + \sin 2\text{C} = 4\sin\text{A}\sin\text{B}\sin\text{C}$.

(2.) $\tan\text{A} + \tan\text{B} + \tan\text{C} = \tan\text{A}\tan\text{B}\tan\text{C}$.

7. Show that, in any triangle, the sides are proportional to the sines of the opposite angles.

Having given

a = 3795 yards, B = 73° 15′ 15″, C = 42° 18′ 30″, find, by help of logarithmic tables, the other sides of the triangle ABC.

8. If R be the radius of the circle circumscribing the triangle ABC, and if p, p', p'' be the distances from the sides of the centre of the circle, show that $2R = \frac{a}{\sin A}$, and that

$$\frac{a}{p} + \frac{b}{p'} + \frac{c}{p''} = \frac{1}{4}\frac{abc}{pp'p''}.$$

9. A statue on the top of a pillar, situated on level ground, is found to subtend the greatest angle (α) at the eye of an observer, when his distance from the pillar is c feet; prove that the length of the statue is $2c \tan \alpha$ feet; and, having given the height above the ground of the observer's eye, find the height of the pillar.

10. Having given the radius (r) of the circle inscribed in a regular polygon of n sides, find the area of the polygon, and hence prove that πr^2 is the area of a circle of radius r.

PURE MATHEMATICS. (1.)

1. If a straight line stand at right angles to each of two straight lines at the point of their intersection, it shall be at right angles to the plane which passes through them.

   If the perpendiculars let fall from the angular points A, D, of a tetrahedron A, B, C, D upon the opposite faces, intersect each other, prove that the edges AD, BC are at right angles to each other.

2. If two straight lines be cut by parallel planes, they shall be cut in the same ratio.

3. Define a sphere, and prove that any plane section of a sphere is a circle.

   If one of the diagonals of a rectangular parallelopiped subtend a right angle at a point, prove that the other diagonals will subtend right angles at the same point.

4. Having given the length of its edge, find the volume of a regular octahedron, and also the volume of its circumscribing sphere.

5. If a, b, c, d be the sides of a quadrilateral which can be inscribed in a circle, and if $2s = a + b + c + d$, prove that the area of the quadrilateral is equal to

$$\sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

6. Assuming De Moivre's theorem for integral indices, prove it for fractional indices.

   By aid of this theorem find all the 5th roots of unity.

7. If n be a positive integer, expand $(\cos\theta)^{2n+1}$ in a series of cosines of multiples of θ.

8. Find the sums of the infinite series:

$$(\alpha)\quad \sin\theta - \frac{\sin 2\theta}{\lfloor 2} + \frac{\sin 3\theta}{\lfloor 3} - \ldots\ldots$$

$$(\beta)\quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots$$

$$(\gamma)\quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots$$

9. Prove that the tangent at any point of a parabola makes equal angles with the focal distance of the point and with the axis.

   Two parabolas have a common focus and axis in the same straight line, but in opposite directions; prove that they intersect each other at right angles.

10. Of two fixed circles, one is entirely within the other; prove that the centre of any circle which can be drawn touching both of the fixed circles lies on one of two ellipses, and that the rectangle contained by the major axes of those ellipses is equal to the difference of the squares on the radii of the fixed circles.

11. Show how to cut from a right circular cone, 1st, a parabola, 2nd, when possible, a rectangular hyperbola.

What is the locus of the foci of all the parabolas which can be cut from a given cone?

12. The axes being rectangular, find the condition that the two straight lines $y = mx + c$, $y = m'x + c'$, may be at right angles to one another.

If (a, b) be the co-ordinates of a point P and (a', b') of a point Q, find the equation of the straight line which bisects PQ at right angles.

13. Find the condition that the line, $y = mx + c$, may be a tangent to the parabola, $y^2 = 4ax$.

Find the locus of the point of intersection of two tangents to a parabola, which are inclined to each other at an angle of 45°.

14. Trace the curve represented by the equation

$$y^3 - 2xy + 2x^3 - 2y + 2x = 0;$$

and find the lengths of its axes.

15. Find the equation of the tangent at the point (x, y) of the curve $xy = a^2$, and prove that every tangent contains with the co-ordinate axes the same area.

PURE MATHEMATICS. (2.)

1. If ABC be a triangle right angled at A, and if ABDE and BCFG be squares upon the side AB and the hypotenuse BC, prove that the lines CD and AG will intersect at right angles.

2. From the point A, one of the angles of the parallelogram ABCD, the line AE is drawn equal in length to the diagonal AC, and upon AE and AD as adjacent sides the parallelogram EADF is completed, prove that if AF be equal to AB, the lines CA and AE will be in the same straight line.

3. From the point O within the triangle ABC the lines OA, OB, and OC, are drawn to the angles of the triangle; prove that if the triangles OAB and OAC be equal to each other, AO produced will bisect BC.

   Hence prove that if a quadrilateral can be divided into four equal triangles by lines drawn to the angles from a point within it, the diagonals of the quadrilateral must bisect each other.

4. If $\frac{x^2}{a} = \frac{y^2}{b} = \frac{1}{c}$ and

   $$a^2y^2(1 - x^2)(1 - b^2) = b^2x^2(1 - y^2)(1 - a^2),$$

   prove that

   $$(1 - a)(1 - b)(1 + c) = (1 + a)(1 + b)(1 - c).$$

   Also if $1 + a + a^2 = 0$,

   prove that $a^3 = 1$, and that

   $$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + ay + a^2z)(x + a^2y + az).$$

5. If the roots of the equation $ax^2 + 2bx + c = 0$ be α and β and those of the equation $Ax^2 + 2Bx + C = 0$ be $\alpha + d$, and $\beta + d$, prove that $\frac{b^2 - ac}{a^2} = \frac{B^2 - AC}{A^2}$.

6. If the second term of an arithmetical progression be a geometric mean between the first and fourth terms, prove that the sixth term will be a geometric mean between the fourth and ninth terms.

7. Find the sum of the series

   $$1 + 2r + 3r^2 + \text{\&c. to infinity.}$$

   Write down the n^{th} term of the series

   $$1 + 3 + 6 + 10 + 15 + 21 + \text{\&c.}$$

   and prove that the sum of n terms is $\frac{n(n+1)(n+2)}{6}$.

8. Given that the arithmetic mean of any number of numbers is always greater than their geometric mean, apply to prove that $2^n - 1 > n\sqrt{2^{n-1}}$.

9. Find the number of the arrangements of the expanded form of the expression $a^m b^2$ such that the two b's come together in each arrangement, and prove that the number of the arrangements in which the two b's never come together is equal to $\frac{m(m+1)}{2}$.

10. If the number of combinations of n things 1 at a time, 2 at a time, &c. n at a time be $c_1, c_2, \ldots c_n$, prove that

$$c_1 + 2c_2 \ldots + nc_n = n2^{n-1}.$$

11. Prove that the coefficient of x^n in the expansion of

$$\frac{1}{1+x+x^2}$$

is 1, 0, or -1 according as n is of the form $3m$, $3m-1$, or $3m+1$.

Prove also that

$$(1+x)^n = 2^n \left\{ 1 - n\frac{1-x}{1+x} + \frac{n(n+1)}{1.2}\left(\frac{1-x}{1+x}\right)^2 - \&c. \right\}$$

12. If $\frac{n^4+n^2-1}{n^3+n^2+n+1}$ be converted into a continued fraction, prove that the quotients will be $n-1$ and $n+1$ alternately, and find the successive convergents.

13. There are 8 volumes, 4 of one work, 3 of another, and 1 of another. If they are placed on a shelf at random, prove that the chance that the volumes of the same works shall all be together is $\frac{3}{140}$.

14. If $3\sin^2 A + 2\sin^2 B = 1$
$3\sin 2A - 2\sin 2B = 0$,

prove that $\sin(B - A) = \frac{1}{9}\sqrt{6}$.

15. If R be the radius of the circumscribing circle, s the semiperimeter, and S the area of the triangle whose sides are a, b, c, prove that

$$\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s} = \frac{4R}{S}.$$

16. Write down the exponential expression for $\cos\theta$.

Prove—

$$2\cos\left(n\cos^{-1}x\right) = \left(x + \sqrt{-1}\sqrt{1-x^2}\right)^n + \left(x - \sqrt{-1}\sqrt{1-x^2}\right)^n.$$

PURE MATHEMATICS. (3.)

1. If $f(x) = x^n + p_1 x^{n-1} + \ldots + p_n$, and if α, β, γ, ... be the roots of the equation $f(x) = 0$, show that

$$f(x) = (x - \alpha)(x - \beta)(x - \gamma) \ldots .$$

Having given that the roots of the equation $x^n - 1 = 0$ are $1\ \alpha, \beta, \gamma, \ldots$. Show that

$$(1 - \alpha)(1 - \beta)(1 - \gamma) \ldots = n.$$

2. State what are the relations between the coefficients and the roots of a rational algebraic equation.

Find the sum of the squares, and also the sum of the cubes, of the roots of the equation

$$x^3 - px - r = 0.$$

3. Prove that an odd number of real roots of the equation $f'(x) = 0$, lies between every adjacent two of the real roots of $f(x) = 0$.

Find the number and position of the real roots of the equation

$$2x^3 - 33x^2 + 180x - 17 = 0.$$

4. Define a differential coefficient, and find from the definition the differential coefficients of

$$\sin x, \log x, \text{ and } \tan^{-1} x.$$

5. Differentiate with regard to x the expressions

$$\frac{x + a}{x^3 + a^3}, \quad \sin^{-1}\frac{x}{\sqrt{1 + x^2}}, \quad \text{and} \quad (\sqrt{1 + x^2})^x.$$

6. If $y = \epsilon^x \sin x$, prove that

$$\frac{d^n y}{dx^n} = 2^{\frac{n}{2}} \cdot \epsilon^x \sin\left(x + n\frac{\pi}{4}\right),$$

and expand y in powers of x by Maclaurin's theorem.

7. If θ be the inclination to the axis of x of the tangent at the point (x, y) of the curve, $y = f(x)$, prove that $\tan\theta = \frac{dy}{dx}$.

Find the points on the curve, $y = x^5 - 5x$, the tangents at which are parallel to the axis of x, and prove that at one of these points the ordinate has a maximum value, and at the other a minimum value.

Also draw a figure of the curve.

8. If $y = (x + \sqrt{1 + x^2})^n$, prove that

$$(1 + x^2)\frac{d^2 y}{dx^2} + x\frac{dy}{dx} = n^2 y,$$

and find the values, when $x = 0$, of

$$\frac{d^2 y}{dx^2}, \quad \frac{d^3 y}{dx^3}, \quad \text{and} \quad \frac{d^4 y}{dx^4}.$$

9. Define the polar subtangent at any point of a curve $r = f(\theta)$, and prove that its length is equal to $r^2 \frac{d\theta}{dr}$.

Find the asymptotes of, and trace the curve, $r = a \tan \theta$.

10. Define the circle of curvature at any point of a curve; and find equations for determining the co-ordinates of the centre of curvature at the point (x, y) of a curve $y = f(x)$.

Prove that, at the point (a, a) of the curve $a^2y = x^3$, the co-ordinates of the centre of curvature are $-4a$, and $\frac{8a}{3}$.

11. Integrate with regard to x the expressions

$$\frac{1}{(x+1)(x+2)}, \quad \frac{x^2}{\sqrt{a^6 - x^6}}, \quad \cot x, \text{ and } \cos x \cdot \cos 2x.$$

12. Prove that

$$\int (a^2 - x^2)^{\frac{r}{2}} dx = \frac{x}{r+1}(a^2 - x^2)^{\frac{r}{2}} + \frac{ra^2}{r+1}\int (a^2 - x^2)^{\frac{r}{2}-1} dx,$$

and find the value of the expression

$$\int_{-a}^{a} \left(a^2 - x^2\right)^{\frac{3}{2}} dx.$$

13. Find the areas enclosed respectively by the curves

(α) $a^2y^2 = x^2(a^2 - x^2)$, (β) $r^2 = a^2 \cos 2\theta$.

STATICS.

1. Assuming the truth of the parallelogram of forces for the direction, prove it also for the magnitude of the resultant.

   A material particle, P, is attracted towards the points A and C, the opposite extremities of the diagonal AC, of the parallelogram ABCD by forces proportional to PA and PC respectively, and it is repelled from the points B and D by forces proportional to PB and PD, prove that the point P will be in equilibrium wherever it be situated.

2. Find the resultant of two parallel forces acting in the same direction.

   Forces proportional to AB, BC, and 2CA act along the sides of the triangle ABC taken in order, find the magnitude and line of action of the resultant force.

3. A string P_1P_2 is kept at rest by equal and opposite forces applied at P_1 and P_2, the resolved parts of these forces parallel to rectangular axes Ox and Oy in the plane of the string being X_1 and X_2, Y_1, and Y_2. Prove that

   $$X_1 + X_2 = 0, \text{ and } Y_1 + Y_2 = 0.$$

   Also if the co-ordinates of P_1 and P_2 be $x_1, y_1, x_2 y_2$, the length of $P_1 P_2$ be l and the tension T, prove that

   $$X_1 x_1 + X_2 x_2 + Y_1 y_1 + Y_2 y_2 = Tl.$$

4. Find the centre of gravity of a triangle.

   If the centre of gravity of the four-sided figure ABCD coincide with one of the angles A, prove that the distances of A and C from the line joining BD are as 1 : 2.

5. State the principal laws of friction and find the angle of inclination at which a body placed upon a rough inclined plane will just begin to slide.

   Two weights are connected by a string and placed upon a rough inclined plane with the string parallel to the line of greatest slope of the plane. If the coefficients of friction between the weights and plane be different, find the angle of inclination of the plane at which they will just begin to slide down, and prove that it is intermediate between the angles of inclination for the weights taken separately.

6. Define the terms "couple" and "axis of a couple," and show how two couples may be compounded into one.

7. Prove that a system of forces in one plane may always be reduced to a force and a couple, and deduce the conditions of equilibrium of such a system of forces acting on a rigid body.

8. A uniform beam AB of given length is supported with its extremity A in contact with a smooth wall by means of a string CD fastened to a known point C of the beam and to a point D

in the wall, if the angle of inclination of the beam to the wall be given, show how to find, by geometrical construction, the length of the string CD and the height of D above A.

Prove that for the problem to be possible the given angle BAD must be acute or obtuse according as AC is less or greater than $\frac{1}{2}$ AB.

9. Find the relation between the power and the weight in the system of pulleys where each hangs by a separate string, and prove that the principle of virtual velocities holds good in this case.

Prove also that this principle holds in the case of a boat propelled by oars; the power being the pull of the rower's arm and the weight the resistance of the water to the boat which may be assumed to be a rectangular trough.

10. State how "work" is measured. A block weighing one ton is in the form of a rectangular parallelopiped 8 feet high on a square base, whose side is 6 feet. It is placed upon a rough board with the sides of its base parallel to the length and breadth of the board, and the centre of the base 6 feet from one extremity of the board.

The board is now tilted round this extremity until the block topples over without sliding; find the work done.

DYNAMICS.

1. Define momentum, force, acceleration. If an accelerating force be represented by 10, when feet and seconds are taken as units, what must the unit of time be in order that the same force may be represented by $67\frac{1}{2}$ when a yard is the unit of length?

2. State the second law of motion; and show that if a body move with velocity v in a given direction its velocity in any direction making angle α with the given direction is $v \cos \alpha$.

   Three equal particles are placed at the angles of the triangle ABC, and move respectively from A to B, B to C, and C to A, the three motions being accomplished with uniform velocity and in the same time; prove that their centre of gravity remains at rest.

3. A body moves with uniform velocity in a circle whose centre is O. Find the velocity and acceleration perpendicular to a given diameter YOY', and show that the motion of the particle relative to YOY' is the same as that of a particle moving in a diameter at right angles to YOY' under the action of a force to O, varying as the distance.

4. A body moves from rest under the action of a constant force. Find the time of describing a given space, and explain the double result.

   If O be a fixed point, C a point in a vertical plane through O, find the locus of C when the time of sliding down CO varies as the length of CO.

   OX is a horizontal line, C a fixed point in it, OY is drawn in any direction in vertical plane through OC. Find by geometrical construction the point in OY from which the time of sliding down to C in a straight line is the least.

5. Prove that the velocity acquired by a body in sliding down a smooth curve in a vertical plane is the same as that acquired in falling perpendicularly through the same vertical height.

   ABC is a circular arc whose middle point is B. A particle is projected from A so as to slide along the curve, and is acted on by a constant force in the plane of the curve. If the velocities at A, B, C respectively be known, show how to find the magnitude and direction of the force.

6. Two particles are simultaneously projected from the same point with given velocities and inclinations to the horizon in different vertical planes; prove that their centre of gravity describes a parabola.

   If one of the vertical planes remain fixed while the other assumes all possible positions, find the locus of the foci of all such parabolas.

7. If a particle be attracted to a centre of force varying as the distance, prove that the time of falling from rest to the centre is independent of the initial distance.

8. Define a cycloid, and prove that the length of the arc measured from the vertex is twice the corresponding chord of the generating circle.

A pendulum of length l has one end of the string fastened to a peg on a smooth plane inclined to the horizon at angle α. With the string and the weight on the plane its time of oscillation is two seconds. Find α, having given that a pendulum of length $\frac{l}{2\sqrt{2}}$ oscillates in one second when suspended vertically.

9. Find the central force under which a particle will move with uniform velocity in a circle.

A particle is projected horizontally with velocity v from the highest point of a smooth vertical circle; find when it will leave the curve.

If a parabola with vertical axis be substituted for the circle, prove that the particle will never leave it unless it leave it at the vertex.

10. Two elastic spheres of masses m_1, m_2, impinge obliquely; v_1, v_2 are their velocities at any time. It being given that $m_1 v_1^2 + m_2 v_2^2$, is the same after impact as before, prove that the velocity of separation is equal to that of approach.

Two unequal perfectly elastic particles capable of moving freely in a smooth circular tube are projected in opposite directions with unequal velocities along the tube from the same point in it. Determine the angular distances from the point of projection of the points of successive impacts.

SET IV.

EUCLID.

1. If a side of a triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.

   A is the greatest angle of a triangle ABC; show how to divide A into two parts whose difference is equal to B.

2. Parallelograms on the same base and between the same parallels are equal to one another.

   On the side of the base BC remote from the vertex A of an acute-angled triangle a rectangle BDEC is described. If AD, AE meet BC at F and G, and AH is drawn parallel to BD, and meeting DE at H, show that the quadrilateral AFHG is equal to the triangle ABC.

3. In any right-angled triangle, the square on the side subtending the right angle is equal to the sum of the squares on the sides containing the right angle.

   Give a geometrical construction for a square three times as great as a given square.

4. If a straight line be divided into two equal, and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line and of the square on the line between the points of section.

5. The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

   Prove that the converse proposition is also true.

   If in any quadrilateral the diagonals are equal to one another, show that the straight lines joining the middle points of opposite sides are at right angles.

6. Show how to draw a straight line touching two given circles.

   Prove that the difference of the squares on the common tangents to two equal circles is equal to the square on a diameter.

   If two equal circles cut at right angles, show that the common tangent is equal to the common chord.

7. About a given circle describe a triangle equiangular to a given triangle.

8. If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be equiangular to one another.

9. Similar triangles are to one another in the duplicate ratio of their homologous sides.

From an isosceles triangle cut off a part bearing to the whole triangle the ratio of the base to a side.

10. If from the vertical angle of a triangle a straight line be drawn perpendicular to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

11. Prove that the circle described through the middle points of the sides of a triangle passes also through the feet of the perpendiculars from the angular points on the opposite sides. Find the centre of this circle, and show that its radius is half that of the circle circumscribing the triangle.

ALGEBRA.

1. Find the product, and the sum of the squares of the expressions

$$ax + by,\ bx - ay,\ ay + bx, \text{ and } by - ax.$$

If $y = 2x$, and $a = 3b$, find the value of

$$\frac{(ax + by)(ay + bx)(a^2 + x^2)}{(bx - ay)(by - ax)(b^2 + y^2)}.$$

2. Divide $x^5 + 8x^4 - 26x^3 + 117x^2 - 220x + 64$ by $x^2 + 11x - 4$.

Also divide $x^3 + y^3 + z^3 - 3xyz$ by $x + y + z$.

3. If $x = a + b + \dfrac{(a-b)^2}{4(a+b)}$ and $y = \dfrac{a+b}{4} + \dfrac{ab}{a+b}$,

prove that $(x - a)^2 - (y - b)^2 = b^2$.

4. Find the Greatest Common Measure of the algebraic expressions

$$x^3 - 12x^2 - 50x + 75, \text{ and } x^3 - 16x^2 + 18x - 45.$$

5. Prove that the sum of the squares of two unequal quantities is greater than twice their product.

For what value of x has the expression $x(1 - x)$ its greatest possible value?

If a, b, c be in harmonic progression, prove that $a^2 + c^2$ is greater than $2b^2$.

6. Find the square roots of

$$x^4 + 4ax^3 + 2a^2x^2 - 4a^3x + a^4,$$

and of $x^6 - 22x^4 + 34x^3 + 121x^2 - 374x + 289$.

7. Solve the equations

$$(\alpha)\ 3x - \frac{x-5}{4} - \frac{x+7}{3} = \frac{x-1}{12} + 27,$$

$$(\beta)\ 12x^2 - 95x + 175 = 0,$$

$$(\gamma)\ \frac{a}{x-a} + \frac{b}{x-b} = \frac{a}{b} + \frac{b}{a},$$

$$(\delta)\ \left.\begin{aligned} x^2 + 3y^2 &= 43 \\ x^2 + xy &= 28 \end{aligned}\right\}.$$

8. The same amount of tax was paid on the profits of a business when the income tax was 6*d.* per £1 as when the profits of the business were £300 more, and the income tax 2*d.* less per £1. Find the profits in each case.

9. A person, A, owes certain sums of money to B and C, and he finds that if he pays B and C sums of money in the ratio of 9 : 17, the sums unpaid will be respectively 10 per cent. and 15 per cent. of the debts, and will together amount to £52. Find the amount of each debt.

10. If $a : b :: c : d$, prove that

$$a(a+b+c+d) = (a+b)(a+c),$$

and that

$$a^3 b^2 : c^3 d^2 :: a^5 + b^5 : c^5 + d^5.$$

11. If α, β be the roots of the equation

$$x^2 - px + q = 0,$$

show that $\alpha + \beta = p$, and $\alpha\beta = q$.

Prove that the equation the roots of which are $\frac{\alpha}{\beta^2}$ and $\frac{\beta}{\alpha^2}$ is

$$q^2x^2 - (p^3 - 3pq)\,x + q = 0.$$

12. Having given the first term and the common difference of an arithmetic series, find the n^{th} term and the sum of n terms. Find also the sum of $2n+1$ terms beginning with the $(n+1)^{\text{th}}$ term.

13. If the quantities a, b, c, d, e, be in geometrical progression, prove that

$$c(a + 2c + e) = (b+d)^2.$$

Find the sum of n terms of the series, the r^{th} term of which is $(2r+1)\,2^r$.

14. Write down the expression for the number of permutations of n things taken r together, and hence deduce the number of combinations of n things taken r together.

Find the value of n when the number of combinations of n things taken 3 together is 12 times the number of combinations of $\frac{n}{2}$ things taken 3 together.

15. Enunciate the binomial theorem, and write down the coefficient of x^4 in the expansions of $(1-x)^{-3}$ and $(2-3x)^{\frac{3}{2}}$.

Prove that the coefficient of x^r in the expansion of $(1-4x)^{-\frac{1}{2}}$ is $\frac{\lfloor 2r}{(\lfloor r)^2}$.

TRIGONOMETRY.

1. Define the circular measure of an angle.

Find the number of degrees subtended at the centre of a circle by an arc = ·357 radius, taking π to be 3·1416.

2. Define the principal trigonometrical ratios, and express each of them in terms of the tangent.

If $1 + \sin^2\theta = 3\sin\theta\cos\theta$, find $\tan\theta$.

3. Write down a formula for all angles having the same sine.

$$\text{If } \frac{\sin^4\theta}{\sin^2 a} + \frac{\cos^4\theta}{\cos^2 a} = 1;$$

prove that the general value of θ is $n\pi \pm a$ where n is any integer.

4. Prove geometrically that

$$\sin(A - B) = \sin A\cos B - \cos A\sin B$$

where A is > B and A and B are each less than 90°.

$$\text{If } a\sin A + b\sin B + c\sin C = 0,$$
$$\text{and } a\cos A + b\cos B + c\cos C = 0,$$

$$\text{prove that } \frac{\sin(B - C)}{a} = \frac{\sin(C - A)}{b} = \frac{\sin(A - B)}{c}$$

5. Find $\sin\frac{A}{2}$ in terms of $\sin A$.

Apply the formula to find sin 15°.

6. Find sin 3A and cos 3A, and prove that

$$\cos 3A - \sin 3A = (\cos A + \sin A)(1 - 2\sin 2A).$$

Prove also that

$$\cos(45° - A)\cos(45° - B) - \cos(45° + A)\cos(45° + B) = \sin(A + B).$$

7. Explain the notation $\tan^{-1} m$.

Given $2\tan^{-1}\cos\theta = \tan^{-1} 2\operatorname{cosec}\theta$, find θ.

8. Prove that in any triangle ABC

$$\tan\frac{A - B}{2} = \frac{a - b}{a + b}\cot\frac{C}{2}.$$

If a = 1·56234, b = ·43766, C = 58° 42′ 6″, find A and B by the aid of the following logarithms:—

$\log 56234 = 4\frac{3}{4}$ $\quad$ $\log\cot 29° 21′ = ·250015$

$\log\cot 29° 22′ = ·249715.$

9. Find an expression for the radius of a circle inscribed in a given triangle.

Prove that if this radius be r, and if p_1, p_2, p_3, be the perpendiculars from the angles upon the opposite sides respectively, then

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}.$$

10. A person standing at a point A, due south of a tower built on a horizontal plane observes the altitude of the tower to be 60°. He then walks to a point B, due west from A, and observes the altitude to be 45°, and again at the point C in AB produced he observes the altitude to be 30°. Prove that B is midway between A and C.

11. If D be the middle point of the side BC of the triangle ABC, and if S be the area of the triangle, prove that

$$\cot \mathrm{ADB} = \frac{\mathrm{AC}^2 - \mathrm{AB}^2}{4\mathrm{S}}.$$

12. Find the area of a regular polygon of n sides, the length of each side being a.

If the length of each side of a regular heptagon be 1 foot, prove, by means of logarithmic tables, that the area of the heptagon in square feet (to 3 places of decimals) will be 3·634, and find the difference between the area of the heptagon and that of the circumscribing circle, taking π to be 3·1416.

PURE MATHEMATICS. (1.)

1. Give a geometrical construction for drawing a perpendicular to a given plane from a given point without it.

2. Show that the shortest distance between two straight lines in different planes is at right angles to each of the lines.

3. Find an expression for the radius of the sphere inscribed in a polyhedron, the volume of the solid and the areas of its faces being known.

   The base of a tetrahedron is equally inclined to the other three faces, which are mutually at right angles: show that the radius of the inscribed sphere is $\frac{h}{1+\sqrt{3}}$ where h is the height of the tetrahedron.

   Find also the radius of the sphere touching the base and the other faces produced.

4. If Δ be the area of a quadrilateral inscribed in a circle, show that the diagonals meet at the angle $\sin^{-1}\frac{2\Delta}{ac+bd}$ where a, b, c, d, are the sides of the quadrilateral.

   If circles can be both inscribed in and described about the quadrilateral, show that the angle between the diagonals is $\cos^{-1}\frac{ac \sim bd}{ac+bd}$.

5. Show that $\tan(\alpha+\beta+\gamma+\ldots) = \frac{S_1 - S_3 + S_5 - \ldots}{1 - S_2 + S_4 - \ldots}$ where S_r denotes the sum of the products of the tangents of the angles $\alpha, \beta, \gamma, \ldots$ taken r together.

6. Sum the series

   (1.) $\operatorname{Sin}\alpha - \operatorname{Sin}(\alpha+\beta) + \operatorname{Sin}(\alpha+2\beta) - \ldots$

   (2.) $\operatorname{Sin}\alpha + \operatorname{Sin}\beta \operatorname{Sin} 2\alpha + \operatorname{Sin}^2\beta \operatorname{Sin} 3\alpha + \ldots$

   each to n terms.

7. Show how to resolve the expression $x^{2n} - 2x^n \operatorname{Cos}\theta + 1$ into quadratic factors.

8. Show that the tangents at the extremities of a focal chord of a parabola intersect on the directrix, and are at right angles.

9. The square on the semi-minor axis of an elliptic section of a right circular cone is equal to the product of the perpendiculars from the extremities of its major axis on the axis of the cone.

   The semi-vertical angle of a cone is 30°; show that the volume cut off by a plane at right angles to one of its generating lines and at a distance a from the vertex is $\frac{\pi a^3}{2\sqrt{6}}$.

10. Find the equation to a straight line in terms of its intercepts on the axes of co-ordinates. Find also the length of the perpendicular on it from a given point.

11. Find the equations to the straight lines bisecting the angles between

$$\frac{x-a}{l} = \frac{y-b}{m}, \quad \frac{x-a}{l'} = \frac{y-b}{m'}.$$

12. Define the eccentricity of a conic section, and show that the eccentricity of a rectangular hyperbola is $\sqrt{2}$.

13. Find the equation to the tangent at any point of an ellipse in terms of its inclination to the axis.

A circle radius r is concentric with an ellipse, whose semi-axes are a, b. Show that the common tangent is inclined to the major axis at the angle $\tan^{-1}\sqrt{\frac{r^2-b^2}{a^2-r^2}}$, and find its length.

14. Find the polar equation to a rectangular hyperbola, the centre being the pole; and show that if r be any radius vector, p the perpendicular on the tangent at its extremity, $rp = a^2$, where a is the semi-axis.

PURE MATHEMATICS. (2.)

1. If two circles intersect, prove that their common chord bisects either of their common tangents.
2. The quadrilateral ABCD is inscribed in a circle, and its diagonals AC and BD intersect each other at right angles in E. Prove that if F be the foot of the perpendicular from E upon AB, and G be the middle point of CD, then EF and EG will be in the same straight line.
3. If BAE and CAD be two straight lines intersecting in A, and if BA have to AE the same ratio that DA has to AC, prove that BD will be parallel to CE.
4. The two planes CAB and DAB intersect in the straight line AB, and the points P and Q are taken in CAB and DAB respectively. From P and Q the lines PR and QR are drawn to a point R in AB. Prove that when R is such that the sum of the distances PR and QR is the least possible, the angle PRA is equal to the angle QRB.
5. Prove that the equation
$$(x-a)^2+(y-b)^2+(a^2+b^2-1)(x^2+y^2-1)=0,$$
is equivalent to the equation
$$(ax+by-1)^2+(bx-ay)^2=0,$$
and therefore that the only possible values of x and y are
$$\frac{a}{a^2+b^2} \text{ and } \frac{b}{a^2+b^2}.$$
6. If α and β be the roots of $x^2+px+1=0$,
and
γ and δ be the roots of $x^2+qx+1=0$,
prove that
$$(\alpha-\gamma)(\beta-\gamma)(\alpha+\delta)(\beta+\delta)=q^2-p^2.$$
7. If there be n positive quantities $a_1, a_2 \ldots\ldots a_n$, and if the square roots of all their products, 2 and 2, as $\sqrt{a_1a_2}$, $\sqrt{a_1a_3}$, &c., be found, prove that the sum $\sqrt{a_1a_2}+\sqrt{a_1a_3}+$ &c. is less than
$$\frac{n-1}{2}(a_1+a_2\ldots\ldots+a_n);$$
hence prove that the arithmetic mean of the square roots of the products 2 and 2, is less than the arithmetic mean of the given magnitudes.
8. If $a+b+c=2s$ and if $s-a$ be the geometric mean of $s-b$ and $s-c$, prove that $2(s-a)$ will be the harmonic mean between b and c.
9. Assuming that 1^2+2^2+ &c. to n terms
$=\frac{n(n+1)}{6}\frac{(2n+1)}{}$, prove that the sum of the harmonic means of all the pairs of positive integers whose sum is equal to m will be
$$\frac{1}{3}(m^2-1).$$

10. Find the sum of the series

$$1+\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+\text{\&c. to } n \text{ terms.}$$

And apply the result to find the sum of n terms of the series whose r^{th} term is $\frac{r(r+1)}{3^r}$.

11. In how many ways may the 5 letters a, b, c, d, e be arranged so that no one of them is removed more than one place at most from its alphabetical order?

12. Given $\log_3 9 = a$ and $\log_3 5 = b$,
Find log 2 and log 3 to base 10 in terms of a and b.

13. If $\frac{p_n}{q_n}$ and $\frac{p_{n-1}}{q_{n-1}}$ be successive convergents to any continued fraction of which the quotients are $m_1, m_2, \ldots\ldots m_n$, &c., and if $\frac{p_n}{p_{n-1}}$ be converted into a continued fraction, prove that the successive quotients in this latter case will be m_n, m_{n-1}, m_{n-2}, &c.

14. If $(1+x+x^2)^{3r} = 1+k_1x+k_2x^2+$ &c.,
and if $(x-1)^{3r} = x^{3r} - c_1x^{3r-1} + c_2x^{3r-2}$, &c.,
prove that
(1.) $1 - k_1 + k_2 -$ &c. $= 1$.

(2.) $1 - k_1c_1 + k_2c_2 -$ &c. $= \pm \frac{\lfloor 3r}{\lfloor r \,\lfloor 2r}$.

15. If $\tan\theta = \frac{\sin\alpha\cos\beta}{\sin\beta+\cos\alpha}$,
prove that

$$\tan\frac{\theta}{2} = \frac{(1-\sin\beta)(1-\cos\alpha)}{\sin\alpha\cos\beta},$$

and that $\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \frac{\cos\left(\frac{\pi}{4}-\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\pi}{4}-\frac{\alpha-\beta}{2}\right)}$.

16. If $2s = a+b+c$,
prove that $(b-c)(s-a)^2 = s^2(b-c) - a(b^2-c^2)$.
And prove that in any triangle ABC whose sides are a, b, and c,

$$(b-c)(s-a)\cos^2\frac{A}{2} + (c-a)(s-b)\cos^2\frac{B}{2} + (a-b)(s-c)\cos^2\frac{C}{2} = 0.$$

17. Prove from De Moivre's theorem or otherwise that

$$\text{Log}\frac{\cos n\theta}{\cos^n\theta} = \frac{n\tan^2\theta - \tan^2 n\theta}{2} - \frac{n\tan^4\theta - \tan^4 n\theta}{4} + \text{\&c.}$$

PURE MATHEMATICS. (3.)

1. If the equation
$$x^3 + px^2 + qx + r = 0$$
be known to have all its roots real, give rules for determining their signs, examining especially the case in which p or q vanishes.

2. If $f(x)$ be an algebraical expression, prove that if the roots of $f(x) = 0$ be all real, those of $f'(x) = 0$ will be all real also.

3. Define a reciprocal equation, and show that its degree may always be reduced to at least one half of its former value. Solve the equation
$$x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0.$$

4. Give general rules for testing the convergency or divergency of series, and examine the case of the series whose n^{th} term is
$$\frac{3n+1}{(n+2)^2}.$$
Explain the importance of the theory of convergency in the finding of differential coefficients.

5. Give a rigid investigation of the differential coefficient of x^n.

6. Investigate and prove Leibnitz's Theorem.
Find $\frac{d^6}{dx^6}(x^4 \log x)$.

7. Prove that, with certain restrictions as to $\phi(x)$, $\phi'(x)$ will vanish for a value of x between any two values which make $\phi(x)$ vanish.
Expand $(1 + x)\tan^2 x$ as far as x^4.

8. Prove that $\frac{d^2u}{dxdy} = \frac{d^2u}{dydx}$, and verify in the case of
$$u = xy^2 \log(1 + \tan xy).$$

9. Prove that if $\phi(x)$ be a maximum or minimum $\phi'(x)$ in general vanishes.
Find the maximum or minimum values of
$3x^4 + 4x^3 - 30x^2 + 36x + 3 = 0.$

10. Give a rule for finding the asymptotes to a curve, and find those of
$y^3 + 2xy^2 - x^2y - 2x^3 + 3x + 4y + 7 = 0.$

11. Investigate an expression for the radius of curvature in rectangular co-ordinates, and find it at the extremity of the major axis of an ellipse.

12. Find the locus of a point such that two of the normals from it to the parabola

$$y^2 = 4ax$$

may be coincident. What is the locus called?

13. Prove the fundamental theorem connecting integration with summation.

Find the integrals of $\sqrt{a^2 - x^2}\, dx$,

$\dfrac{x dx}{(x+1)(x+2)}$, and $xe^x\, dx$.

14. Investigate the formula for the rectification of a curve in polar co-ordinates, and find the polar equation to the curve for which $s = cr$.

STATICS.

1. Show that forces can be represented by straight lines.

   Define the resultant of two forces acting at a point, and explain the use of a parallelogram in determining this resultant.

   A material particle P is attracted to three points A, B, C by forces proportional to the distances PA, PB, PC; prove that the resultant of these forces is in direction of the centre of gravity G of the triangle ABC, and is proportional to the distance PG.

2. If three forces acting in one plane balance each other, prove that their lines of action are either parallel or meet in a point.

   A heavy rod AB is supported, in a position inclined 60° to the vertical, by a string OA fastened to a fixed point O, and by a horizontal force acting at the lower end B; find the inclination of the string OA to the vertical.

3. A string passing over two smooth parallel bars in the same horizontal plane, has two equal weights fastened to its ends, and another equal weight is fastened to a point of the string between the bars; find the position of equilibrium of the system, and the pressure upon each bar.

4. Neglecting the weights of the pulleys, find the ratio of the power to the weight in the system of pulleys in which each hangs by a separate string.

   If the power be moved through a given space, find the height through which the weight will be lifted.

5. A smooth sphere is supported in contact with a smooth vertical wall by a string fastened to a point on its surface, the other end being fastened to a point on the wall; if the length of the string be equal to the radius of the sphere, find the position of equilibrium, the tension of the string, and the pressure on the wall.

6. If two forces act at a point, prove that the moment of their resultant about any point in the plane containing their lines of action is equal to the sum of the moments of the two forces about the same point.

   Three forces act along the sides of a triangle; prove that, if the sum of two of the forces be equal to the third force, the resultant of the three forces will pass through the centre of the inscribed circle of the triangle.

7. What is the direction of the action between—

   1st. Two smooth surfaces in contact.

   2nd. An edge and a smooth surface with which the edge is in contact.

   3rd. Two edges in contact with each other.

   A solid sphere rests upon two parallel bars in the same horizontal plane, the distance between the bars being equal to the radius of the sphere; find the pressure on each bar.

8. Find the moment of a couple which is equivalent to two given couples acting in the same plane.

Forces act along the sides of a plane polygon, taken in order, and their intensities are proportional to the respective sides along which they act; prove that this system of forces is equivalent to a couple, the moment of which is proportional to the area of the polygon.

9. Having given the positions of the centres of gravity of the whole of a body, and of a given portion of it, show how to find the centre of gravity of the remainder.

A solid, in the form of a right circular cone, has its base scooped out, so that the hollow formed is a right cone on the same base and of half the height of the original cone; find the position of the centre of gravity of the solid so formed.

10. A flat circular disc of given weight is lying on a rough horizontal plane; find the magnitude and direction of the least force acting at its centre by which it can be moved.

A solid cylinder rests on a rough horizontal plane with one of its flat ends on the plane, and is acted upon by a horizontal force through the centre of its upper end; if this force be just sufficient to move the solid, prove that it will begin to slide, or to turn over, according as the coefficient of friction is less or greater than the ratio of the radius to the height.

11. Explain what is meant by work, and define a foot-pound, and a horse-power.

A weight of 10 tons is dragged, in half an hour, a length of 330 feet up a rough plane inclined 30° to the horizontal plane; the coefficient of friction being $\frac{1}{\sqrt{3}}$, find the work expended, and the horse-power of an engine by which the work could be done.

12. State the principle of work (i. e. the principle of virtual velocities) as applied to any system of equilibrating forces in one plane.

A square figure ABCD is formed by four equal rods jointed together, and the system is suspended from the joint A, and kept in the form of a square by a string connecting A and C; prove that the tension of the string is half the weight of the four rods, and find the direction and magnitude of the action at either of the joints B or D.

13. State the relation between the tension and the extension of an elastic string.

Prove that the work expended in pulling out an elastic string to any assigned length is equal to half the extension multiplied by the final tension.

DYNAMICS.

1. Explain how velocity is measured, (1) when uniform, (2) when variable.

   Two straight lines of railway meet at right angles. A train starts from the junction on one line, and at the same instant another train starts towards the junction from a station on the other line, and they move at the same given speed. Find their distance apart at any time, and prove that they are nearest to each other when they are equally distant from the junction.

2. Explain how velocities are resolved and compounded.

   A railway train is moving at the rate of 28 miles an hour when a pistol shot strikes it horizontally in a direction making the angle $\sin^{-1}\frac{3}{5}$ with the train.

   The shot enters a compartment of one of the carriages at the corner farthest from the engine, and passes out at the diagonally opposite corner, the compartment being 8 feet long and 6 wide. Prove that the shot is moving at the rate of 80 miles an hour, and that it traverses the carriage in $\frac{5}{44}$ of a second.

3. A body is projected with the velocity v, and moves for the time t with the uniform acceleration f. Find the space described. Prove that this space is the same as would be described by a body moving during the same time with a uniform velocity equal to the arithmetic mean of the initial and final velocities.

4. Apply the second law of motion to prove that the path of a projectile in vacuo is a parabola.

   If balls be fired at the same instant from two cannons with equal velocities at angles of elevation, α and β respectively, so that both hit the same mark, and if t be the time between their hitting the mark, and t' the time between their returning to the horizontal plane through the point of projection, prove that

$$t' = 2t\cos^2\frac{\alpha+\beta}{2}.$$

5. Bodies slide down a number of smooth inclined planes of the same height. Prove that the velocities acquired in the descent are the same for all the bodies, but that the times of descent are proportional to the cosecants of the inclinations of the planes to the horizon.

6. Find the acceleration when two unequal weights are connected by a string passing over a smooth pulley.

   If the pulley be rough, and if the effect of friction be to prevent motion until the tension of the string at one end be greater than the tension at the other by $\frac{1}{n}$th part of the latter tension, prove

that the effect on the acceleration will be the same as if the pulley remained smooth and the smaller weight were increased by $\frac{1}{n}$th of itself.

7. Find the velocities of two given elastic spheres after direct impact.

A ball at rest on a smooth horizontal plane at the distance of one yard from a wall, is impinged on directly by another equal ball moving at right angles to the wall with the velocity of one yard in a minute.

If the coefficients of elasticity between the balls, and between the balls and wall, be $\frac{1}{2}$; prove that the balls will impinge a second time after 2′ 24″, the radii of the balls being of inconsiderable length.

8. Find the magnitude and direction of the acceleration of a particle describing the circumference of a circle with uniform velocity.

Hence explain what is meant by *centrifugal force*.

A particle is suspended by a string from the roof of a railway carriage which is moving with velocity v, on a curve of radius r; find the inclination of the string to the vertical, when it is at rest, relatively to the carriage.

9. Prove that the length of any arc of a cycloid measured from the vertex is equal to twice the corresponding chord of the generating circle; and find the resolved part of gravity in the direction of the tangent at any point of the arc of a cycloid whose vertex is downwards and base horizontal.

10. Given that, when a particle is placed at any distance from a fixed point towards which it is attracted by a force always equal to μr (where r is the distance of the particle at any instant from that point), the time of reaching the point is $\frac{\pi}{2\sqrt{\mu}}$; apply the result of the last question to find the time of oscillation in an inverted cycloid, and deduce the time of a small oscillation of an ordinary pendulum.

A clock which at the surface of the earth at a certain place gains 10″ a day, loses 10″ a day when taken down a mine; compare the force of gravity at the surface and bottom of the mine.

SET V.

EUCLID.

1. If two triangles have two sides of the one equal to two sides of the other, but the angle contained by these two sides in the one greater than the angle contained by the two sides equal to them in the other, then shall the base of the former be greater than the base of the latter.

2. Define a parallelogram, and prove that if the opposite sides or opposite angles of a quadrilateral figure be equal, the figure will be a parallelogram.

3. Describe a square equal to a given rectilineal figure.

4. In Euclid I. 47, if the sides of the square on the side subtending the right angle be produced to meet the sides, and sides produced, of the other squares, show that two of the triangles so formed are equiangular and equal to the given triangle.

5. If two circles touch each other internally, the straight line which joins their centres being produced shall pass through the point of contact.

6. Upon a given straight line describe a segment of a circle, which shall contain an angle equal to a given rectilineal angle.

   If from any point without a circle lines be drawn touching it, the angle contained by the tangents is double the angle contained by the line joining the points of contact and the diameter drawn through one of them.

7. Describe a circle about a given triangle. The radii of the circles described about two triangles, which have one side equal to one side, and the angles opposite to this side either equal or supplementary to one another are equal.

8. Describe a circle about a given equilateral and equiangular pentagon. If the sides of the pentagon be produced to meet, the angles formed by these lines are together equal to two right angles.

9. If the sides of two triangles about each of their angles be proportionals, the triangles shall be equiangular, and the equal angles shall be those which are opposite to the homologous sides.

10. Equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides.

11. If semicircles be described on the segments of the base made by a perpendicular drawn from the right angle of a triangle, they will cut off from the sides, segments which will be in the triplicate ratio of the sides.

ALGEBRA.

1. If $a = \frac{25}{16}$, $b = 1$, $c = \frac{3}{4}$,

prove that

$$(a - \sqrt{b})(\sqrt{a} + b)\sqrt{a - b} = \frac{8c^4}{\sqrt{a - c^2}};$$

and, if $2s = a + b + c$,

prove that

$$3abc = s^3 - (s - a)^3 - (s - b)^3 - (s - c)^3.$$

2. Define a^m, when m is a positive integer, and if n also be a positive integer, prove that $a^m \times a^n = a^{m+n}$.

Assuming that this equation is true, when m and n are negative or fractional, deduce the interpretations of $a^{\frac{1}{n}}$ and of a^{-n}.

Simplify the expression

$$\sqrt[3]{a^6 b^{\frac{2}{3}} c^{\frac{1}{2}}} \times b^{\frac{7}{9}} \times \left(c^{\frac{1}{3}} a^2\right)^{-\frac{1}{2}}$$

3. Divide $1 - 5x + \frac{152}{15}x^3 - \frac{106}{225}x^4 - \frac{28}{9}x^5$

by $1 - x - \frac{14}{15}x^2$.

Resolve into two simple factors the expression

$$x^2 - 19x + 84,$$

and resolve into two quadratic factors the expression

$$x^4 + a^2x^2 + a^4.$$

4. Prove the rule for finding the Greatest Common Measure of two algebraical expressions.

Find the G.C.M. of

$$7x^4 - 10ax^3 + 3a^2x^2 - 4a^3x + 4a^4.$$
$$\text{and } 8x^4 - 13ax^3 + 5a^2x^2 - 3a^3x + 3a^4.$$

5. If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, prove that each of these fractions

$$= \frac{x + y + z}{a + b + c} = \sqrt[3]{\frac{x^3 + y^3 + z^3}{a^3 + b^3 + c^3}}$$

If $\frac{y + z}{ay + bz} = \frac{z + x}{az + bx} = \frac{x + y}{ax + by}$,

prove that each of these fractions $= \frac{2}{a + b}$.

6. If $\sqrt{a+\sqrt{b}} = x+\sqrt{y}$,

prove that $\sqrt{a-\sqrt{b}} = x-\sqrt{y}$.

Find the square root of $280+56\sqrt{21}$.

7. Solve the equations

(α.) $\dfrac{4x^3+4x^2+8x+1}{2x^2+2x+3} = \dfrac{2x^2+2x+1}{x+1}$.

(β.) $\dfrac{12}{x}+\dfrac{10}{x-1} = 6\frac{1}{3}$.

(γ.) $\left.\begin{matrix} x^4+x^2y^2+y^4 = 243 \\ x^2+xy+y^2 = 9 \end{matrix}\right\}$.

8. Find the condition that the roots of the equation $ax^2+bx+c=0$, may be (1) real and unequal, (2) equal, (3) imaginary.

Form the equation the roots of which are

$$\frac{1}{2}(4 \pm \sqrt{7}).$$

9. Having given the first term (a) and the n^{th} term (l) of an arithmetic series, find the common difference and the sum of n terms.

Find the sums of the series of decimals;

(α.) $5+6\,.\,2+7\,.\,4+8\,.\,6+\ldots\ldots$ to 21 terms.

(β.) $5+1+\,.\,2+\,.\,04+\ldots\ldots$ to infinity.

10. Write down the expressions for the number of permutations of n things taken all together, and for the number of combinations of n things taken r together.

If N denote the number of combinations of n things 2 together, prove that the number of combinations of N things 2 together is three times the number of combinations of $n+1$ things 4 together.

Of 12 men, 2 can steer and cannot row, and the rest can row but cannot steer: in how many ways can the crew of an eight-oar, with a coxswain, be made up?

11. Write down the coefficients of x^r in the expansions of $(1+x)^n$, $(1-x)^{-2}$, and $(1-x)^{-\frac{1}{2}}$.

If $p_r = \dfrac{1.3.5.\ldots\ldots(2r-1)}{2.4.6.\ldots\ldots\ 2r}$, prove that

$$p_n+p_{n-1}\cdot p_1+p_{n-2}\cdot p_2+\ldots\ldots+p_1p_{n-1}+p_n = 1.$$

12. Two vessels contain mixtures of wine and water; in one there is three times as much wine as water, in the other five times as much water as wine; find how much must be drawn off from each to fill a third vessel which holds seven gallons, in order that its contents may be half wine and half water.

13. A ship did a measured mile in four minutes less on a perfectly calm day under steam than on a windy day under sail; also on the windy day, using steam in addition to sail, she did the mile in $2\frac{2}{3}$ minutes; what was the ship's time over the mile under steam on the calm day?

TRIGONOMETRY.

1. Find the circular measure of an angle of A degrees.

Find the radius of a globe such that the distance measured upon its surface between two places on the same meridian whose latitudes differ by 1° 10′ may be one inch, assuming the unit of circular measure to be $\frac{7}{11}$ of a right angle.

2. Trace the changes in sign and magnitude of cos A through the four quadrants, and deduce a general expression for all angles having the same cosine.

If $\cos p\theta + \cos q\theta = 0$,
prove that the different values of θ form two arithmetical progressions in which the common differences are

$$\frac{2\pi}{p+q} \text{ and } \frac{2\pi}{p \backsim q} \text{ respectively.}$$

3. Prove that

$$\sin (A + B) = \sin A \cos B + \cos A \sin B,$$

pointing out any restrictions in the values of A and B assumed in your proof.

If (1) $x \cos \beta + y \cos \alpha = z$
(2) $x \sin \beta - y \sin \alpha = 0$

Prove that $\frac{x}{\sin \alpha} = \frac{y}{\sin \beta} = \frac{z}{\sin (\alpha + \beta)}$.

4. Prove that $\sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}}$,

and determine the limits to the value of A in order that the lower sign may be taken.

Prove that

(1.) $\cos^2 A \cos^2 B + \sin^2 A \sin^2 B = \frac{1 + \cos 2A \cos 2B}{2}$.

(2.) $\tan 2A = (1 + \sec 2A) \tan A$.

5. Find sin 3A and apply the result to determine sin 18°.

6. Explain the notation $\tan^{-1} m$.

If $\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = \pi$
Prove that $\alpha + \beta + \gamma = \alpha\beta\gamma$.

7. In any triangle ABC whose sides are a b, and c, prove that sin A : sin B : sin C :: a : b : c.

If in such a triangle

$$\frac{\tan A - \tan B}{\tan A + \tan B} = \frac{c - b}{c},$$

Prove that A = 60°.

8. Given two sides and the difference of the angles opposite to them; solve the triangle.

If the sides be 4 and 5 and the cosine of the difference of the opposite angles be $\frac{31}{32}$, prove that the third side will be 6.

9. Find the radius of the circle circumscribing a given triangle.

If R and r be the radii of the circumscribing and inscribed circles and a, b, and c the sides of the triangle, prove that

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{1}{2Rr}.$$

10. If $a = 3$, $b = 1$, $C = 53° 7' 48''$, find c, without determining A and B, with the aid of the following logarithms:—

log 2 = ·3010300.
log 25298 = 4·4030862.
log 25299 = 4·4031034.
L cos 26° 33′ 54″ = 9·9515452.
L tan 26° 33′ 54″ = 9·6989700.

11. Find the area of a regular polygon of n sides inscribed in a circle of radius R.

Find by the aid of logarithmic tables the area of a regular polygon of 11 sides inscribed in a circle of radius one foot to 3 places of decimals.

PURE MATHEMATICS. (1.)

1. Show that planes to which the same straight line is perpendicular are parallel to one another.

2. Given a plane and two points on the same side of it, find a point in the plane such that the sum of its distances from the two points is the least.

3. Show that the section of a sphere made by any plane is a circle.

   How many circles can be drawn to pass through two points on the surface of a sphere so as to be concentric with it?

4. Given a right circular cone in which two spheres are inscribed so as to touch a plane of section on opposite sides at S and H. Show that if P be any point on the line of section SP + HP is constant.

5. If s be the sum of the sides of a triangle, a and A any side and the opposite angle, the area of the triangle is $s(s-a)\tan\frac{A}{2}$.

   If a regular octagon be described about a circle, and a similar octagon be inscribed in the circle, the area of the described figure is to that of the inscribed figure as $2\sqrt{2}(\sqrt{2}-1) : 1$.

6. Show that $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \&c.$

   The quantity $\sin^{-1}x + \cos^{-1}x$ is independent of x.

7. Sum to $2n$ terms the series

   $$\cos\alpha + \cos(\alpha + 2\beta) + \cos(\alpha + 4\beta) + \ldots.$$

8. Two poles, a and b feet long respectively, are placed vertically on a horizontal plane so as to subtend the angle α at a point in the line joining their feet. If β, β', be the angles which they subtend at any point in the horizontal plane at which the line joining their feet subtends a right angle, then

   $$(a+b)^2\cot^2\alpha = a^2\cot^2\beta + b^2\cot^2\beta'.$$

9. Find the polar equation to a straight line passing through two given points.

   A triangle whose right angle is at C has its side AB fixed. A point P is taken on AC such that $AP^2 - CP^2 = BP^2$. What is the locus of P?

10. Find the locus of a point which so moves that the difference of its distances from two fixed lines at right angles is equal to its distance from a fixed line.

11. Find the equation of a tangent to a circle in terms of one variable.

    Given a chord of constant length in a given circle, find the locus of intersections of tangents drawn at its extremities.

12. Define an ellipse, and from your definition find its equation in the form $a^2y^2 + b^2x^2 = a^2b^2$; show also that if one diameter of an ellipse bisect all chords parallel to a second, the second diameter will bisect all chords parallel to the first.

13. What is an equilateral hyperbola?

Given two concentric equilateral hyperbolas such that their axes include an angle of 45°. Show that they cut each other at right angles.

14. Find the equation to the normal at $(x'\ y')$ to the parabola whose equation is $y^2 = 4ax$.

Express this equation in terms of the tangent of the angle which the normal makes with the axis of x; and thence show that three normals can be drawn to a parabola from a point not on the curve.

15. Show that in the parabola $p^2 = ar$, where p is the perpendicular from the focus on the tangent at any point, and r the radius vector from the focus to the same point.

PURE MATHEMATICS. (2.)

1. If AEB be a triangle, and if C, D be points in BE, AE such that the angle EDC is equal to the angle EBA, prove that a circle can be described about the quadrilateral ABCD.

2. If $(a^2 + b^2)(x^2 + y^2) = (ax + by)^2$, prove that $\frac{x}{a} = \frac{y}{b}$; and if $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax + by + cz)^2$, prove that $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.

3. The hour and minute hands of a clock are at right angles to each other at 3 o'clock; when are they next at right angles to each other?

4. Prove that
$$(x + y + z)^3 - x^3 - y^3 - z^3 = 3(y + z)(z + x)(x + y).$$
Prove also that $(x + y + z)^5 - x^5 - y^5 - z^5$ is divisible by $(y + z)(z + x)(x + y)$, and find the other factor.

5. The interior angles of an irregular polygon are in arithmetic progression; the least angle is 120° and the common difference 5°; find the number of sides.

6. O is a fixed point, and P is a point which moves on a fixed straight line; in the straight line OP a point Q is taken, such that the rectangle OQ, OP is constant; prove that the locus of Q is a circle.

7. OA, OB, OC are three fixed straight lines, such that the angles AOB, BOC are equal, and from a fixed point P perpendiculars PD, PE, PF are let fall upon OA, OB, and OC respectively; prove that the straight lines DE, EF are equal.

8. Prove that, if n be any positive integer,
$n(n + 1)(2n + 1)$ is a multiple of 6.

9. On a railway there are 15 stations; find how many tickets are required in order to travel from any one station to any other.

10. Find the least number which, when expressed in the scale of which 3 is the base, has zero for its three right-hand digits; when in the scale of which 4 is the base, has two zero digits on the right; and when in the scale of which 5 is the base, has one zero digit on the right. Express this number in the duodecimal scale.

11. If $\sin \alpha = \tan \beta$, prove that
$$\sin(\alpha - \beta) \cos \beta = \sin 2\beta \sin^2 \frac{\alpha}{2}.$$

12. Having given
$$\log_{10} 2 = \cdot 301030 \text{ and } \log_{10} 7 = \cdot 845098,$$
find $\log_{10} 98$ and $\log_{1000} \frac{343}{8}$.

13. If the distance between two stations C and D on a horizontal plane subtend equal angles (α) at two other stations A and B on the plane, and if s be the sum of the angles ABD, BAC, prove that

$$CD = AB \frac{\sin \alpha}{\sin (s - \alpha)}.$$

14. Find the relation between the sides of a triangle ABC, when

(1) $\text{Sin B sin C} = \cos^2 \frac{A}{2}$.

(2) $\text{Sin B sin C} = \sin^2 \frac{A}{2}$.

15. The tangents at two points A and B of a circle intersect in T; prove that if from any point P of the circle, perpendiculars PL, PM, PN be let fall upon the chord AB, and on the tangents TA, TB respectively, PL is a mean proportional between PM and PN.

16. A regular octahedron is inscribed in a cube so that the corners of it are at the centres of the faces of the cube; show that the volume of the cube is six times the volume of the octahedron.

17. Two numbers a and b being given, two others a_1 and b_1 are formed by the relations

$$3a_1 = 2a + b, \; 3b_1 = a + 2b,$$

two more a_2, b_2 are formed from a_1, b_1, in the same manner and so on: find a_n and b_n in terms of a and b, and prove that, if n increase indefinitely, a_n and b_n are ultimately equal.

18. If $\cos (\theta + \phi\sqrt{-1}) = \cos \alpha + \sqrt{-1} \sin \alpha$, prove

$$\tan^2\theta - \tan^2\alpha = \pm \tan \alpha \sec \alpha.$$

PURE MATHEMATICS. (3.)

1. Prove that a rational algebraic equation of an even degree, the last term of which is negative, has at least two real roots, one positive and the other negative.

   What is the nature of the roots of the equation

   $$x^4 + x^2 + 7x + 13 = 0?$$

2. State the relations between the coefficients and the roots of the rational algebraic equation

   $$x_n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n = 0.$$

   Find the sum of the reciprocals of the roots of this equation, and also the sum of the squares of the reciprocals, in terms of the coefficients.

3. Prove that in any rational equation the number of positive roots cannot exceed the number of changes in the signs of the coefficients.

   Show that the equation

   $$x^6 - x^4 + x^3 + x + 1 = 0,$$

   has at least two imaginary roots, and find the least possible number of imaginary roots of the equation

   $$x^9 - x^5 + x^4 + x^2 + 1 = 0.$$

4. Define the differential coefficient, with regard to x, of a function of x; and find, from the definition, the differential coefficients with regard to x, of

   (1) $x^3 + ax^2 + a^3$, (2) $\sin x$, (3) $\log x$.

5. Differentiate with regard to x the expressions

   $$\log(\sin x), \quad \sin(\log x), \quad \text{and} \quad \left(\frac{1 - x + x^2}{1 + x + x^2}\right)^{\frac{3}{2}}.$$

6. If $y = (x + \sqrt{1 + x^2})^r$, prove that

   $$(1 + x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx} = r^2 y,$$

   and find the values, when $x = 0$, of $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$.

7. If θ be the inclination to the axis of x of the tangent at the point (x, y) of a curve, prove that $\tan\theta = \frac{dy}{dx}$.

   Find the points on the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, at which the tangent is equally inclined to the axes, and prove that the tangent at any one of these points makes, with the axes, a triangle the area of which is $\frac{1}{2}(a^2 + b^2)$.

8. If $f(x) = 0$ be a rational algebraic equation, prove that an odd number of the real roots of the equation, $f'(x) = 0$, lie between every two consecutive real roots of the equation, $f(x) = 0$.

Illustrate, or prove, this theorem by geometrical considerations.

Find the number and position of the real roots of the equation

$$x^3 - 6x^2 + 9x + 15 = 0.$$

9. Find the maxima and minima values of the expressions

(1) $3x - x^3$,

(2) $\dfrac{\cos x}{2 + \sin x}$.

10. State Maclaurin's theorem for the expansion of $f(x)$ in powers of x.

If $y = e^x \cos x$, prove that

$$\frac{d^r y}{dx^r} = 2^{\frac{r}{2}} e^x \cos\left(x + r\frac{\pi}{4}\right),$$

and expand y in powers of x, writing down the n^{th} term of the series.

11. Define an asymptote of a curve, and find the asymptotes of the curve

$$y(x^2 - a^2) = x^3 - 8a^3.$$

Prove that the curve intersects one of its asymptotes at the distance $8a\sqrt{2}$ from the origin.

12. Define the circle of curvature at any point of a curve, and show how to find its radius, and the co-ordinates of its centre.

Prove that the radius of curvature at the point $\left(a, \frac{a}{4}\right)$ of the curve $4a^2y = x^3$, is $\dfrac{125}{96}a$.

13. Write down the integrals, with regard to x, of

$\cos x$, $x \cos(x^2)$, $\dfrac{1}{x^2 + a^2}$, and $\dfrac{2x}{x^2 + a^2}$.

Find the integrals of

$$x \cos x, \quad \frac{1}{x^4 - a^4}, \text{ and } \frac{\cos x}{\sin^2 x + \sin^2 a}.$$

14. Prove that the area of a curve referred to polar co-ordinates is equal to

$$\frac{1}{2}\int r^2 d\theta.$$

Trace the curve $r^2 = a^2 \sin 2\theta$, and find its area.

STATICS.

1. Assuming the parallelogram of forces for the direction, prove it for the magnitude of the resultant.

   If O be the centre of the circle circumscribed about the triangle ABC, and if OD, OE, OF be the perpendiculars from O upon the sides of the triangle, prove that the six forces represented by AO, BO, CO, OD, OE, and OF are in equilibrium.

2. When two parallel forces acting on a rigid body in opposite directions admit of a single resultant, prove that the moment of that resultant about any point in the plane of the forces is equal to the algebraic sum of the moments of the forces about the same point.

   Prove also that when the forces do not admit of a single resultant the algebraic sum of their moments about every point in the plane of the forces is the same, wherever the point may be situated.

3. Prove that every rigid body has one and only one centre of gravity.

   A body consists of two parts, A and B, whose centres of gravity are at the points P and Q respectively. The part B is moved so that its centre of gravity comes to Q′, the part A remaining fixed, prove that the centre of gravity of the whole body moves through a distance parallel and proportional to QQ′.

4. By what laws are the magnitude and direction of friction determined when the point at which the friction acts is in a state bordering on motion?

   A heavy particle P is placed upon a rough plane inclined at the angle α to the horizon, and is connected by the stretched weightless string AP to the fixed point A in the plane.

   If AB be the line of greatest slope in the plane through A and θ the angle PAB when the particle is on the point of slipping, prove that

   $$\sin\theta = \mu \cot\alpha,$$

   and interpret the result when $\mu \cot\alpha$ is greater than unity.

5. Find the relation between the power and the weight in that system of pulleys in which each string is attached to the weight, the weights of the pulleys being neglected.

   Two weights each of 1 lb. support each other by means of a fine string passing over a pulley without weight, and to this pulley another string is attached which passes over a fixed pulley and supports a weight of 2 lbs.

   If one of the 1 lb. weights be gently raised through any height, prove that the sum of the products of each of the

remaining weights into the distance through which it sinks is equal to the product of the weight raised into the height through which it is raised.

6. Prove that every system of forces acting on a rigid body in one plane may be reduced either to a single force or a single couple, and deduce the conditions of equilibrium.

7. A uniform heavy rod AB is suspended horizontally by two equal weightless strings, CA and DB, attached to the nails C and D respectively in a vertical wall, the lengths CD and AB being equal.

A cone whose weight is half that of the rod, slant side longer than either string and vertical angle 30°, is thrust, with its vertex downwards, between the rod and the wall touching the former in the middle point, and the latter along a vertical line.

Prove that when there is equilibrium the inclination of each string to the wall will be 30°. [No friction.]

8. Show how to graduate the common steelyard.

A weighing machine is constructed so that for each complete stone placed in the weighing-pan an additional weight of m oz. has to be suspended at the end of the arm, one foot in length, while the odd pounds in the weighing-pan are measured by a weight of n oz. sliding along the arm.

Prove that the distance between the graduations of the arm for successive lbs. must be $\frac{6m}{7n}$ inches.

9. Find the relation between the power and the weight in the screw, and illustrate thereby the principle of virtual velocities.

10. What is the measure of *work* done by an agent? and what is meant by *horse power*?

A rectangular parallelopiped, 3 feet broad, 6 feet long and 8 feet high, and weighing half a ton, rests upon a horizontal plane.

Find the work required to be done in turning it over round the shorter side of the base, and the time in which it can be thus turned over by an engine of one horse power.

DYNAMICS.

1. Explain the terms momentum, accelerating force. Two inelastic spheres moving in opposite directions impinge directly on each other. Find their velocities after impact.

   Six inelastic balls of equal mass are placed not in contact in a row. Another inelastic ball of equal mass moving with a velocity v impinges directly on the first ball in the direction of the row. Find what velocity will be acquired by the last ball.

2. A body moves from rest under the influence of a uniformly accelerating force. Show that the spaces described in the n^{th} and $n-1^{th}$ units of time are as $2n-1 : 2n-3$.

   A stone is thrown down a well with a velocity of 64 feet per second, and strikes the surface of the water in 6 seconds. If the force of gravity be 32 feet per second, and if the sound reaches the thrower at once, find the depth of the well.

3. A heavy elastic ball drops from the ceiling of a room, and after twice rebounding from the floor just reaches a point half the height of the room. Show that its elasticity is $\sqrt[4]{\frac{1}{2}}$.

4. Show that if two smooth balls impinge directly on each other the motion of their centre of gravity is unchanged by the impact.

5. The velocity of a projectile moving in vacuo at any point of its path is equal to that which would be acquired by a heavy particle falling freely down a space equal to the vertical distance of that point from the directrix of the curve described by the projectile.

   A heavy elastic ball falls from a height of n feet upon a plane inclined at an angle of 60° to the horizon. Find the distance between the first two points at which it will strike the plane.

6. A heavy body is drawn up a smooth inclined plane by another body which descends vertically. The bodies are connected by a fine inextensible string. Find the tension of the string at any time.

7. A heavy particle moves down a semicircle placed in a vertical plane with its base horizontal from rest. Show that the velocity at the lowest point varies as the square root of the radius of the semicircle.

8. The diameter of the earth, regarded as a sphere, is approximately 8000 miles. A seconds pendulum loses 20 beats per day when taken to the top of a mountain. Find its height.

9. State and prove the principle known as the parallelogram of velocities.

A man fires at an object moving with a given velocity in a given direction. The velocity of the ball is such that it would have hit the object, if at rest, in one second. How far ahead of its apparent position must he aim to hit the object?

10. Show that the effect of centrifugal force is to lessen the weight of a body m at the latitude λ by $\frac{mg\cos^2\lambda}{289}$, approximately, the earth being assumed to be a sphere whose diameter is 8000 miles.

11. Two weights, 3P and P, move on two planes each inclined to the horizon at an angle of 45°, and are connected by a fine string which passes over their common vertex.

Find the acceleration of their centre of gravity. What is its locus?

12. Explain the terms potential energy, kinetic energy.

Given a smooth tube bent into the form of n equal inverted V's: the tube is held in a vertical plane, with the summits of the inverted V's in a horizontal line; show that if a particle be projected up the tube from one end with a velocity greater than that due to falling under the influence of gravity through a distance equal to the height of the V, it will emerge from the tube with unchanged velocity.

SET VI.

EUCLID.

1. Construct a triangle of which the sides shall be equal to three given straight lines. What condition is necessary? Show how the construction would otherwise fail. Construct also a triangle, of which one angle is given, an adjacent side, and the difference of the other two sides.

2. If a side of any triangle be produced, prove that the exterior angle is equal to the two interior and opposite angles, and that the three interior angles of every triangle are equal to two right angles.

   State and prove also the corollaries of this proposition.

3. Describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

   The squares of the diagonals of a parallelogram are together equal to the squares of the four sides.

4. In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle is greater than the squares of the sides containing the obtuse angle by twice the rectangle contained by the side upon which, when produced, the perpendicular falls and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.

   If one angle of a triangle be equal to two-thirds of two right angles, prove that the square of the side subtending it is equal to the sum of the squares of the sides containing it, together with the rectangle contained by those sides.

5. The opposite angles of a quadrilateral figure inscribed in a circle are together equal to two right angles.

   If two sides of this quadrilateral be produced to meet, the triangles so formed are equiangular.

6. If from any point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it, the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square of the line which touches it.

   AB is a chord and AC a diameter of a circle. AC is produced to D and ED drawn perpendicular to it. If AB produced intersects ED in E, prove that the rectangle BA, AE is equal to the rectangle CA, AD.

7. Inscribe an equilateral and equiangular hexagon in a given circle.

Compare the magnitudes of an equilateral triangle, a square, and hexagon in the same circle, whose radius is unity.

8. What are the loci of the following points?

(1.) The vertices of equal triangles on the same base.

(2.) The vertices of triangles on the same base with equal vertical angles.

(3.) The vertices of isosceles triangles on the same base.

(4.) The centres of equal straight lines in a circle.

9. When is the first of four magnitudes said to have a greater ratio to the second than the third has to the fourth? When is a straight line said to be cut in extreme and mean ratio?

Define alternando, componendo, and ex æquali.

10. If four straight lines be proportionals, the similar rectilineal figures similarly described upon them shall also be proportionals.

11. In equal circles angles, whether at the centres or circumferences, have the same ratio as the circumferences on which they stand.

If the diameter of a semicircle be divided into any number of parts, and on them semicircles be described, their circumferences will together be equal to the circumference of the given semicircle.

ALGEBRA.

1. Simplify

$$\left[\frac{a^2+x^2}{a^2-x^2}-\frac{a^2+x^2}{a^2-x^2}\right] \div \left[\frac{a^2-x^2}{a^2+x^2}-\frac{a-x}{a+x}\right]$$

and show that if $x^3 - 3b^2x + 2c^3$ is divisible by $x - a$, and $x - b$, then either $a = b = c$

or $-a = 2b = 2c$.

2. If $a + b + c = 0$, then

$$a\frac{b^3-c^3}{b-c}+b\frac{c^3-a^3}{c-a}+c\frac{a^3-b^3}{a-b}=0.$$

3. Show that if $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, then each of these fractions is equal to

$$\sqrt{\frac{lx^2+my^2+nz^2}{la^2+mb^2+nc^2}},$$

where l m n are any quantities.

4. Find the G.C.M. of

$3x^4 - 4x^3 + 1$ and $4x^4 - 5x^3 - x^2 + x + 1$,

and the L.C.M. of

$a^2 + b^2$, $a^2 - b^2$, $a^3 + a^2b + ab^2 + b^3$,

and $a^3 - a^2b + ab^2 - b^3$.

5. Solve the equations

i. $\frac{\sqrt{x+2}+\sqrt{x-2}}{\sqrt{x+2}-\sqrt{x-2}}=\frac{7+3\sqrt{5}}{2}$.

ii. $x^2y = 2z$; $y^2z = 9x$; $xyz = 6$.

iii. $\frac{a}{b}x^2+\left[1+\frac{b}{a}\right]x-\frac{1}{2}=\frac{b}{a}x^2-\left[1-\frac{a}{b}\right]x+\frac{1}{2}$.

6. Prove that if $x^2 - px + q = 0$ has equal roots, $p^2 = 4q$.

If the equation $x^2 + 2(1 + k)x + k^2 = 0$ has equal roots, what must be the value of k?

7. If $\sqrt{q}$ and $\sqrt{y}$ are incommensurable quantities, and if

$$\sqrt[r]{p+\sqrt{q}} = x + \sqrt{y},$$

then prove that

$$\sqrt[r]{p-\sqrt{q}} = x - \sqrt{y}.$$

Also find the square root of $16 + 6\sqrt{7}$.

8. Show how to transform a number from the scale whose radix is (r_1) to that whose radix is (r_2). Transform 25603 from the nonary to the septenary scale.

9. Define Geometrical and Harmonical Progression.

If a b and c are in harmonical progression, show that

$$\left[\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right]\left[\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right]=\frac{4}{ac}-\frac{3}{b^2}.$$

The $n-1^{th}$ and $n+1^{th}$ terms of a geometrical series are the arithmetical and harmonical means between two quantities. Show that the n^{th} term is the geometrical mean between the same two quantities.

10. Given the sum of n terms of an arithmetical progression, a the first term, and b the common difference; find the number of the terms.

Sum the series

$$\frac{2}{3}+3+\frac{16}{3}+\frac{23}{3}+\ldots\text{to 16 terms},$$

and show that

$$1+2+3+4+\ldots+n=[1^3+2^3+3^3+\ldots+n^3]^{\frac{1}{2}}.$$

11. Show that the number of combinations of n things taken r together is equal to that of n things taken $n-r$ together.

If ${}_nC_r$ is the number of combinations of n things taken r together, show that

$$n\times {}_{n-1}C_{r-1}=r\times {}_nC_r.$$

How many words of 4 consonants and 1 vowel can be made out of 12 consonants and 5 vowels?

12. If c_r is the coefficient of x^r in the expansion of $(1+x)^n$, write down the values of c_1, c_2, c_3, c_4, and c_0, and show that

$$c_0+c_2+c_4+\ldots\&c.=c_1+c_3+c_5+\ldots\&c.$$

Find the middle term in the expansion of $\left[5a-\frac{x}{5}\right]^{16}$.

13. A train A starts to go from P to Q, two stations 240 miles apart, and travels uniformly. An hour later another train B starts from P and after travelling for two hours comes to a point that A had passed 45 minutes previously. The pace of B is now increased by 5 miles an hour; and it overtakes A just on entering Q. Find the rates at which they started.

TRIGONOMETRY.

1. Show that the angle subtended at the centre of a circle by an arc equal to the radius is invariable.

   Taking 22 : 7 as the ratio of the circumference of a circle to its diameter, and assuming the diameter of the earth to be 8000 miles, find approximately the difference in latitude of two places, one of which is 100 miles north of the other.

2. Find an expression for $\sin \frac{A}{2}$ in terms (1) of sin A, (2) of cos A, and explain why more than one value appears.

   Find sin 15° and cos 165°.

3. Show that

   (1.) $\cot A - \tan A = 2 \cot 2A$.

   (2.) $\tan (A + B) \tan (A - B) = \frac{\sin^2 A - \sin^2 B}{\cos^2 A - \sin^2 B}$.

4. Find an expression for all the angles which have a given tangent.

   Show that $\tan^{-1}(\cot A) - \tan^{-1}(\tan A) = n\pi + \frac{\pi}{2} - 2A$.

5. Solve the equation $\tan^2\theta = 3 \operatorname{cosec}^2\theta - 1$.

6. In a triangle ABC, A is a right angle and B = 60°.

   Show that

   $2a = c + b\sqrt{3}$.

7. Show that in every triangle

   (1.) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$.

   (2.) $\frac{a \cos B - b \cos A}{c} = \frac{\sin^2 A - \sin^2 B}{\sin^2 C}$.

8. Find an expression for the area of a triangle.

   In an isosceles right-angled triangle a straight line is drawn from the middle point of one of the equal sides to the opposite angle. Show that it divides the angle into parts whose cotangents are 2 and 3.

9. Find an expression for the radius of the circle inscribed in a given triangle.

   Two sides of a triangle are a and $2a$, and they include a right angle. If r_1 r_2 be the radii of the escribed circles touching these sides, show that $r_1 r_2 = a^2$.

10. Two sides and the angle opposite to one of them being given, explain by means of a figure why more than one triangle may in some cases be found having the given parts.

In a triangle $a = 5$, $b = 4$, $A = 45°$. Find the other angles, having given

$\log 2 = \cdot 3010300$, L sin 33° 29′ = 9·7520507.

L sin 33° 30′ = 9·7530993.

11. If θ be the acute angle between two adjacent sides of a rhombus, show that

$$\theta = \cos^{-1} \frac{x^2 \backsim y^2}{x^2 + y^2},$$

where x, y are the lengths of the diagonals.

PURE MATHEMATICS. (1.)

1. When is a straight line perpendicular to a plane, and when is a plane perpendicular to a plane?

   Prove that if three straight lines meet in one point, and a straight line stand at right angles to each of them at that point, the three straight lines shall be in one and the same plane.

2. Define a solid angle. If a solid angle be contained by three plane angles, prove that the three plane angles together are less than four right angles.

   State the nature and the number of the plane angles that form a solid angle in each of the five regular solids.

3. What is a great circle of a sphere? A plane passes through the centre of a sphere and three points on its surface equidistant from each other; how must the three points be situated?

   Prove that two great circles of a sphere are bisected at their points of intersection.

4. If a right cone with a circular base be cut by a plane parallel to a slant side of the cone and perpendicular to the plane passing through that slant side and the axis of the cone, prove geometrically that the curve of intersection is a parabola.

5. Regular polygons of the same number of sides are inscribed in and circumscribed about a given circle. Compare the areas of the polygons, and determine the number of sides when the area of the inscribed polygon is $\frac{3}{4}$ the area of the circumscribed polygon.

6. If $2\cos\theta = x + \frac{1}{x}$, prove $2\cos m\theta = x^m + \frac{1}{x^m}$.

   Prove also, if m be a whole number, $2^{m-1}(\cos\theta)^m =$

   $$\cos m\theta + m\cos(m-2)\theta + \frac{m.m-1}{1.2}\cos(m-4)\theta + \&c.;$$

   write down the last term of this series when m is odd.

7. If r be the radius of a circle and πr^2 its area, what ratio does the constant π express?

   Prove $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$; point out, without performing the calculation, how the numerical value of π may be thus determined.

8. Two towers whose heights respectively are 180 and 80 feet stand on a horizontal plane; from the foot of each column the angle of elevation of the other is taken, and one angle is found to be double the other; prove that the horizontal distance between the towers is 240 feet, and show that the sine of the greater angle of elevation is $\frac{3}{5}$.

9. The equation to a straight line passing through a point (x_1y_1) is $y - y_1 = m(x - x_1)$. What does m represent (1) when the coordinate axes are rectangular, (2) oblique?

The axes being rectangular, find the perpendicular from the origin on the straight line whose equation is

$$4(x - 4) = 3(3 - y).$$

10. Construct the circle whose equation is

$$16x^2 + 16y^2 - 16x + 8y - 139 = 0;$$

and express the equation to this circle referred to rectangular axes with the centre of the circle for the origin.

11. Find the rectangular equation to the tangent at any point of the parabola whose equation is $y^2 = 4ax$.

Find the equation for determining the points in which the straight line $y = mx + c$ cuts the parabola, and show that the straight line becomes a tangent if $c = \frac{a}{m}$.

12. Find the polar equation to an ellipse referred to a focus.

If r and r_1 are the focal distances of the extremities of a chord passing through the focus, prove that $\frac{1}{r} + \frac{1}{r_1}$ is constant.

13. Define the asymptote of a curve; find the equation to an hyperbola referred to its asymptotes as axes. What does the equation become if the hyperbola is rectangular?

It being given that $yx_1 + xy_1 = a^2$, is the equation to the tangent to a rectangular hyperbola drawn at a point x_1y_1 and referred to the asymptotes as axes, prove that the area of the triangle contained between the tangent at any point and the asymptotes is constant.

PURE MATHEMATICS. (2.)

1. Prove that

$$x(y+z)^2 + y(z+x)^2 + z(x+y)^2 - 4xyz = (y+z)(z+x)(x+y).$$

2. Solve the equations

$$\left.\begin{aligned}(z+x)(x+y) &= 9\\ (x+y)(y+z) &= 16\\ (y+z)(z+x) &= 25\end{aligned}\right\}$$

3. A person started at a certain pace to walk to a railway station five miles off, allowing himself just time to catch a train; after walking one mile, he stopped for twenty minutes, but, by walking one mile an hour faster, he was still in time for the train; at what pace did he start?

4. If a quadrilateral be inscribed in a circle, prove that the sum of the angles in the four segments of the circle exterior to the quadrilateral is equal to six right angles.

5. Describe a circle touching a given circle and a given straight line, and having its centre on another given straight line.

6. Solve the equation

$$\sin 5x \cos 8x = \sin 9x \cos 7x;$$

and reduce $\cos(\theta + \phi\sqrt{-1})$ to the form $\alpha + \beta\sqrt{-1}$.

7. If D in BC be the point of contact of the circle inscribed in a triangle ABC, and E the point of contact of the circle which touches BC and the other two sides produced, prove that DE is equal to the difference between AB and AC.

8. If a circle touch two fixed circles, prove that the straight line joining the points of contact will pass through one of two fixed points.

9. If a be the arithmetic mean between b and c, and b the geometric mean between a and c, prove that c will be the harmonic mean between a and b.

10. Show that if n be an even integer,

$$\frac{1}{\lfloor 1 \rfloor n-1} + \frac{1}{\lfloor 3 \rfloor n-3} + \frac{1}{\lfloor 5 \rfloor n-5} + \ldots\ldots + \frac{1}{\lfloor n-1 \rfloor 1} = \frac{2^{n-1}}{\lfloor n}.$$

11. Find a number of six digits, having 7 in the unit's place, such that if it be multiplied by 3 the product is the same as the number obtained by transferring the first digit on the left from the left to the right of the number.

12. Prove that

$$2^6 (\cos^8 a + \sin^8 a) = (\cos 8a + 28 \cos 4a + 35),$$

and find the values of θ between 0 and $\frac{\pi}{2}$ which satisfy the equation

$$\sin^2\theta + \cos^2 2\theta = \frac{3}{4}.$$

13. What are the logarithms to base 3 of 81, $\frac{1}{81}$, and $\sqrt[4]{729}$? How many positive integers are there whose logarithms to base 3 have 2 for a characteristic?

14. Three equal circles touch each other. Find the radius of the circle circumscribing the three circles.

15. A solid cone, the vertical angle of which is $2 \sin^{-1} \frac{3}{5}$, rolls on a table; find after how many revolutions on the table the point on the rim of the base which is at first in contact with the table will again be in contact with the same point of the table.

16. AD, BE, CF are the perpendiculars from the angular points of a triangle ABC upon the opposite sides; prove that the sides of the triangle DEF are respectively equal to

$$a \cos A, b \cos B, \text{ and } c \cos C.$$

17. A man stands on the top of a wall of height h, and observes the angular elevation (α) of the top of a telegraph post; he then descends from the wall, and finds that the angular elevation is now (β); prove that the height of the post exceeds the height of the man by $h \frac{\sin \beta \cos \alpha}{\sin (\beta - \alpha)}$.

18. If the perpendiculars let fall upon the opposite faces from two angular points, A, B, of a tetrahedron ABCD, intersect each other, prove that the edges AB, CD are perpendicular to each other.

PURE MATHEMATICS. (3.)

1. In a rational algebraic equation, prove that irrational and imaginary roots occur in pairs.

   Form the equation of the fourth degree, of which one root is $\sqrt{2} + \sqrt{-3}$.

2. Show how to transform an equation into another, the roots of which shall be less than the roots of the given equation by a given quantity.

   If $f(x) = x^n + p_1 x^{n-1} + \ldots + p_{n-1}x + p_n$, and if q_n, $q_{n-1}, \ldots q_1$ be the successive remainders when $f(x)$ is divided by $x - a$, the quotient by $x - a$, the next quotient by $x - a$, and so on, prove that the roots of the equation

   $$x^n + q_1 x^{n-1} + \ldots + q_{n-1}x + q_n = 0,$$

   will be less by a than the roots of $f(x) = 0$.

   Diminish by 3 the roots of the equation

   $$x^5 - 4x^4 + 3x^2 - 4x + 6 = 0.$$

3. Explain Cardan's method of solving a cubic equation.

   Prove that the real root of the equation

   $$x^3 + 12x - 12 = 0 \text{ is } 2\sqrt[3]{2} - (\sqrt[3]{2})^2.$$

4. Prove the rule for differentiating the product of two functions.

   Differentiate with regard to x the expressions

   $(x - a)^2 (x - 2a)$, and $\sin x \cos 2x \tan 3x$.

5. Show how to determine, and to distinguish between the maxima and minima values of a function of one variable.

   Find the maxima and minima values of the expression $(x - a)^2 (x - 2a)$, and illustrate this case geometrically by tracing the curve

   $$a^2 y = (x - a)^2 (x - 2a).$$

6. From any point P of a circle, PN is drawn perpendicular to a fixed diameter AOB, and NE is the perpendicular from N upon the tangent at P; the point O being the centre, find the angle POB when the area of the triangle PEN is the greatest possible.

7. Differentiate with regard to x the expressions

   $$\tan^{-1} \frac{\sin x}{1 + \cos x}, \text{ and } \log (\epsilon^x \cos x).$$

   If $y = \dfrac{x^2}{1 \times 2} - \dfrac{x^3}{2 \times 3} + \dfrac{x^4}{3 \times 4} - \dfrac{x^5}{4 \times 5} + \ldots$ to infinity,

   prove that $\dfrac{d^2y}{dx^2} = \dfrac{1}{1 + x}$.

8. If $\phi(a) = 0$, and $\psi(a) = 0$, show how to find the limiting value, when $x = a$, of the fraction $\frac{\phi(x)}{\psi(x)}$.

Find the limiting values of

$$\frac{\tan(a+\theta) - \tan(a-\theta)}{\sin(a+\theta) - \sin(a-\theta)}, \text{ when } \theta = 0.$$

9. Find the equation of the tangent at the point (x, y) of the curve $u = f(x, y) = 0$.

Find the length of the perpendicular from the origin upon the tangent at any point of the curve $xy = c^2$, and prove that the locus of the feet of these perpendiculars is the curve

$$(x^2 + y^2)^2 = 4c^2xy.$$

10. If ψ be the angle between the radius vector and the tangent at the point (r, θ) of the curve $r = f(\theta)$, prove that $\tan\psi = \frac{rd\theta}{dr}$.

Show that the curves

$$r^2\cos(2\theta - \alpha) = a^2 \text{ and } r^2\cos(2\theta - \beta) = b^2,$$

intersect each other at an angle which is independent of a and b.

11. Show how to find the envelope of the system of curves, $f(x, y, a) = 0$, a being a variable parameter.

Find the envelope of the circles described on the double ordinates of a parabola as diameters.

12. Explain the method of integration by parts, and integrate the expressions

$$x^2 \sin x, \text{ and } e^x \sin x.$$

13. Find the integrals of

$$\frac{1}{(x+1)(x+2)(x+3)}, \text{ and } \frac{1+\log x}{x \log x}.$$

14. Prove that the area of a surface of revolution is given by the expression $\int 2\pi y ds$.

Find the area of the surface of a sphere of radius a contained between two parallel planes at a distance c from each other.

STATICS.

1. Define the terms mass, density, weight.

   If the weights of two spherical balls are as 4 : 3, and their densities as 9 : 2, compare their diameters.

2. The system of forces represented in magnitude and direction by the straight lines drawn from any point to the angles of a triangle is equivalent to the system of forces represented by straight lines drawn from the same point to the middle points of the sides.

3. Show how to find the resultant of any number of forces acting in one plane at one point in it.

   Forces of 2, $\sqrt{3}$, 5, $\sqrt{3}$, 2 lbs. respectively, act at one of the angular points of a hexagon towards the five others. Find the direction and magnitude of their resultant.

   Two forces act at a point in a parabola in the directions of and proportional to those portions of the tangent and normal at that point intercepted between the point and the axis of the parabola. Prove that the resultant force passes through the focus.

4. The algebraic sum of the moments of two forces acting in one plane about any point in it is equal to the moment of their resultant about the point. Prove this proposition in the case where the directions of the two forces are not parallel.

   A square uniform plate ABCD rests with its side BC on a perfectly rough plane inclined at an angle (a) to the horizon. A string AP is attached to A the highest point of the plate, and passing over a smooth pulley at P, the top of the plane, supports a weight w. AP is horizontal. If W is the weight of the plate, show that as w increases, it will begin to turn about B when

$$\text{W becomes} < \frac{2w}{1 + \tan a}.$$

5. AB is a smooth plane inclined at an angle (a) to the horizon, and at A the lower end is a hinge about which there works without friction a heavy uniform smooth plank AC, of length $2a$. Between the plane and the plank is placed a smooth cylinder of radius r, which is prevented from sliding down the plane by the pressure of the plank on it from above. If W is the weight of the plank, M the mass of the cylinder, and θ the angle between the plane and the plank, show that in equilibrium

$$\frac{Mgr}{aW} = \cos(a + \theta)\frac{[1 - \cos\theta]}{\sin a}.$$

6. A hollow cylinder, composed of thin metal, open at both ends, of radius a and height $4a$, is placed on a smooth horizontal plane. Inside it are placed two smooth spheres of radius r, one above the other, $2r$ being $> a$ and less than $2a$. If W be the weight of the cylinder, and M the mass of one of the spheres, show

that the cylinder will just stand upright without toppling over if

$$W = 2Mg\frac{a-r}{a}.$$

7. In a combination of n wheels and axles, in which the rope passing round the axle of one machine also passes round the wheel of the next, find the relation between the power and the weight.

   A capstan in the form of a truncated cone of length h feet, with circular extremities of radii r_1 r_2 inches, is placed with its axis horizontal. A cable $2q$ inches in diameter is wound upon it beginning from the smaller end r_1, and its free end hanging vertically supports a weight W. If there be n coils wound upon the capstan, find what power (P) must act at a distance l feet from the axis of the capstan in order to support the weight. Find also the amount of work required to wind these n coils; each coil being regarded as circular, and the weight of the rope being neglected.

8. Explain what is meant by the "angle of friction" and the "coefficient of friction."

   Two inclined planes have a common vertex, and a string passing over a small smooth pulley at the vertex supports two equal weights. If one of the planes be rough and the other smooth, find the relation between the two angles of inclination of the two planes when the weight on the smooth plane is on the point of moving down.

   A heavy rod rests with its extremities on a rough circular hoop fixed in a vertical plane. If the rod subtend an angle of 90° at the centre of the hoop, find its inclination to the horizon when in equilibrium.

9. Show how to find the centre of gravity of any number of particles in one plane.

   Find the centre of gravity of three weights placed at the centres of the circles escribed to a triangle, and inversely proportional to the radii of those circles.

10. Explain the difference between "stable" and "unstable" equilibrium.

    A heavy uniform cube balances on the highest point of a fixed sphere of radius r. Supposing the roughness of the surfaces sufficient to prevent sliding, and that the side of the cube is equal to $\frac{\pi r}{2}$, show that the cube may be made to rock through a right angle without falling.

11. Enunciate the principle of virtual velocities.

    A heavy uniform rod, whose weight is W and length $2a$, revolves freely in a vertical plane about a hinge at one extremity. At a height h above the hinge is a smooth pulley, over which passes a string, one end of which is attached to the rod by a smooth ring that slides freely on the rod, and to the other end is fastened a weight P. Apply the principle of Virtual Velocities to find the position of equilibrium.

DYNAMICS.

1. Define a uniformly retarding force: give three examples of such a force.

   A ball is projected vertically upwards with a velocity V, and describes a space s in a time t; prove $s = Vt - \frac{g}{2}t^2$, where g is the measure of gravity. What value of t gives the greatest value of s for the ascent?

   If V be 200 feet per second, find the time in which the ball after having reached its greatest height will be 600 feet above the point of projection in its descent.

2. If a particle slide from rest down a smooth inclined plane, show that the velocity acquired at the foot of the plane is the same as it would be if it had fallen through the vertical height of the plane: determine also the time of sliding down the plane when its length is given.

   Two vertical circles touch each other internally at A, their highest point; from A any common chord ABC is drawn cutting the inner circle in B and the outer circle in C; show that if a particle slide from A down ABC considered as an inclined plane, the time of describing BC will be the same for all positions of ABC.

3. If a body descend a rough inclined plane whose angle of inclination is α, the limiting angle of resistance being ϕ, show that the accelerating force on the body is equal to $g \sin \frac{(\alpha - \phi)}{\cos \phi}$.

   A body just rests on a plane whose inclination is 30°; find the velocity acquired by a body in sliding from rest down the length of the same plane when it is lifted so that the angle of inclination is 60°, and its vertical height is 150 feet.

4. Find the horizontal range and greatest height of a projectile in vacuo.

   Determine the velocity and angle of projection of a projectile so that it may strike a given mark just when it is moving horizontally.

5. State the third law of motion, and refer to any experiments that confirm the truth of this law.

   A weight of 224 pounds is placed on a smooth table, a uniform pressure acting upon it parallel to the table for five seconds generates in the weight a velocity of 20 feet per second; find the statical measure of this uniform pressure.

6. When a given elastic ball impinges obliquely on a hard plane, given the velocity before impact and the angle of incidence,

determine the velocity after impact and the angle of reflexion. What is meant by perfect elasticity?

A projectile of elasticity e is projected from a point in a horizontal plane at a given angle with the horizon; after rebounding from the plane it strikes it again; if R be the first range, and R_1 the second, prove $R_1 = eR$.

7. Distinguish between a finite and an impulsive force. An elastic ball of given mass moving with a given velocity impinges directly upon another given ball at rest; find the velocity of each after impact, the elasticity also being given.

A ball m impinges upon m_1 at rest, the velocity of m after impact is $\frac{3}{5}$ths of its velocity before impact, and the modulus of elasticity is $\frac{3}{5}$; compare (1) the masses of the two balls, (2) the velocities of m and m_1 after impact.

8. A weight P hanging over the edge of a smooth table descends and draws another weight W along the table by means of an inextensible string; find the accelerating force and the tension of the string.

The tension of the string when P draws 2W is to the tension when P draws W along the table as 3 to 2; compare the weights P and W.

9. If a body whose mass is m be attached to an inextensible string and revolve uniformly round the other end of the string, which is fixed, with a velocity v, the length of the string being r, show that the tension of the string is equal to $\frac{mv^2}{r}$.

If the tension of the string be 9 times the weight of the revolving body, and the length of the string be 2 feet, find the velocity.

10. A particle slides on the circumference from the highest point of a vertical circle; find where it will leave the curve.

11. What is a simple pendulum? Find the time of a small oscillation of a simple pendulum in a circular arc.

How may the force of gravity be ascertained from the motion of the pendulum?

12. When work is done by any agent, how is a unit of work estimated? If a weight W move with a velocity V, determine the number of units of work accumulated in the moving body.

A railway train is moving on a horizontal road at the rate of 30 miles an hour, when the steam is suddenly turned off; how far will the train run before it stops, the resistances being estimated at 7 pounds per ton?

SET VII.

EUCLID.

1. Define a plane angle, a scalene triangle, a rectangle, the angle in a segment of a circle, duplicate ratio.

2. If two triangles have two sides of the one equal to two sides of the other each to each, and have likewise the angles included by those sides equal to one another, they shall also have their bases or third sides equal; and the two triangles shall be equal and their other angles shall be equal, each to each, namely those to which the equal sides are opposite.

3. The opposite sides as well as the opposite angles of a parallelogram are equal to one another, and the diameter bisects it.

   If the two diameters are drawn, show that they bisect each other, and that the parallelogram will be divided into four equal parts. In what case will the diameter bisect the angle of a parallelogram?

4. Triangles on equal bases and between the same parallels are equal.

   Through a point G exterior to the triangle ABC, GD, GE are drawn parallel to CA, CB respectively, and through A and B, AD, BE are drawn parallel to CG, meeting GD, GE in D and E. Show that DE is equal to AB.

5. In any right-angled triangle, the square which is described on the side subtending the right angle is equal to the sum of the squares described on the sides which contain the right angle.

   If ABC be a triangle, right angled at A, and BE, CF be drawn bisecting the opposite sides, then four times the sum of the squares on BE, CF is equal to five times the square on BC.

6. If a straight line be bisected and produced to any point, the square on the whole line thus produced, and the square on the part of it produced, are together double of the square on half the line bisected and of the square on the line made up of the half and the part produced.

7. The angles in the same segment of a circle are equal to one another.

   A straight line AB is placed in a circle whose centre is C. Produce AB to D, making BD equal to the radius of the circle. Draw DECF meeting the circumference in E and F. The arc AF shall be treble the arc BE.

8. If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles which this straight line makes with the line touching the circle shall be equal to the angles which are in the alternate segments of the circle.

   Two circles intersect in the points A and B. A straight line CAD is drawn terminated by the circles: through C and D tangents are drawn to the circles, intersecting in E. Prove that B, C, D, and E lie on the circumference of a circle.

9. Inscribe an equilateral and equiangular hexagon in a given circle.

   If AB, BC are sides of the hexagon, prove that the square on AC is three times the square on AB.

10. If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles are similar.

    If two chords AB, AC drawn from any point A in the circumference of the circle ABC, be produced to meet the tangent at the other extremity of the diameter through A in D and E, then the triangle AED is similar to ABC.

11. If the vertical angle of a triangle be bisected by a straight line which likewise cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square on the straight line which bisects the angle.

ALGEBRA.

1. Multiply together $(x + a - b)$, $(x + b - c)$, $(x + c - a)$, collecting in brackets the coefficients of the powers of x.

   If $a - b = b - c$, express the coefficients of the product in terms of (a) and (b) only.

2. Divide $x^3 - 3a^2x - 2a^3$ by $(x + a)^2$, and hence find a positive algebraical quantity which substituted for (x) shall make $x^3 - 3a^2x - 2a^3 = 0$.

3. Prove the rule for finding the Greatest Common Measure of two numbers (a) and (b).

   Express by four trinomial factors the Least Common Multiple of

   $$(x + 2y)^2 - 9z^2,\ (x + 3z)^2 - 4y^2,\ (2y + 3z)^2 - x^2.$$

   Simplify

   $$\frac{x^2 - (2y - 3z)^2}{(x + 3z)^2 - 4y^2} + \frac{4y^2 - (x - 3z)^2}{(x + 2y)^2 - 9z^2} + \frac{9z^2 - (x - 2y)^2}{(2y + 3z)^2 - x^2}.$$

4. Solve the following equations:—

   (1.) $$\frac{7x - 75}{7x + 75} = \frac{x - 5}{x + 25}.$$

   (2.) $$\left.\begin{aligned} \frac{3(x - 1) + 5y}{2(y + 1) + x} &= \frac{3x}{2y} \\ 5y - 3x &= 3. \end{aligned}\right\}$$

   (3.) $$\left.\begin{aligned} \frac{x}{5} - \frac{y}{7} &= 2 \\ \frac{x}{25} + \frac{y^2}{49} &= 74. \end{aligned}\right\}$$

5. A broker bought as many railway shares as cost him £1875, he reserved 15 of the shares, and sold the remainder for £1740, gaining £4 a share on their cost price; how many shares did he buy, and what price did he pay for each?

6. Show that the square root of an imaginary expression $a + \beta\sqrt{-1}$ may be expressed in a similar form when $a^2 + \beta^2$ is a perfect square.

   Find the square root of $-7 - 24\sqrt{-1}$, and prove it equal to $(1 + 2\sqrt{-1})^2$.

7. If the roots of the equation $x^2 - Px + Q = 0$ be respectively the cubes of the roots of the equations $x^2 - px + q = 0$, find P and Q in terms of the coefficients of this last equation.

   Find the equation whose roots are each the cubes of the roots of the equation $x^2 - 5x + 6 = 0$, and test the accuracy of the result by the solution of the two equations.

8. If (r) be the radix of any scale of notation, examine the relation that must exist among the digits in order that a whole number expressed in that scale may be divisible by $(r+1)$.

Find the radix of the scale in which the whole number 6629 will be expressed by 50405, and divide 50405 in that scale by its radix increased by unity.

9. When are a series of numbers in arithmetical progression? Without assuming the ordinary formula for the sum of the series, find an expression for the sum of (n) terms of the series $a+b$, $a+3b$, $a+5b$, &c., and apply the result to find the sum of 12 terms of the series $6+14+22+$ &c.; prove that the squares of x^2-4x+2, x^2-2x+2, and x^2-2 are in arithmetic progression; verify this by giving (x) any positive value in whole numbers.

10. The sum of three numbers in harmonical progression is 13, the sum of their squares 61; find the numbers.

11. Assuming the expression for determining the number of combinations of (n) things taken (r) together, show that when (n) is an odd number, the number of combinations is the same when

$$r \text{ is } \frac{n+1}{2} \text{ or } \frac{n-1}{2};$$

show also that for either of these values the number of combinations is greater than for any other value of r.

Out of a company of soldiers, combinations of 3 were to be selected, and it was found that the number of different combinations was 1027 times the number of men in the company; what was that number?

12. In the expansion of the binomial theorem show that where the index is a positive integer the number of terms is limited, and where a negative integer unlimited.

Express the $(r+1)^{\text{th}}$ term of $(1-x)^{-n}$.

Compare the coefficient of the 7th term of the expansion of $(1+x)^{n+2}$ with the coefficient of the 7th term of the expansion of $\frac{1}{(1-x)^{n-2}}$, and find (n) when the greater of these coefficients is double the less.

TRIGONOMETRY.

1. Show how to express (1) in grades, (2) in circular measure, an angle whose magnitude is given in degrees.

   The angles of a triangle are in the ratio 1 : 2 : 3. Express their magnitudes in each of the three systems of angular measurement.

2. Show that $\sin(180° - A) = \sin A$,
   and that $\cos(180° - A) = -\cos A$.

   If $\cot\frac{A}{2} = 1 + \sqrt{2}$, find $\cos A$.

3. Establish the formula
   $\text{Sin}(A + B) = \sin A \,.\, \cos B + \cos A \,.\, \sin B$, and show that
   $$\text{Sin } A + \sin B = 2 \sin\frac{A+B}{2} \cos\frac{A-B}{2}.$$

4. Show that

   (1.) $\text{Cosec } A = \cot A + \tan\frac{A}{2}$.

   (2.) $1 = \text{cosec}^2 A - \cot^2 A$.

   (3.) $\text{Tan } A = (1 + \sec A)\tan\frac{A}{2}$.

5. Show that $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$;

   and that $2\cot^{-1} x = \text{cosec}^{-1}\frac{1+x^2}{2x}$.

6. Solve the equation $\text{cosec}\,\theta + \cot\theta = \sqrt{3}$.

7. If A, B, C be the angles of a triangle, and a, b, c the sides opposite to them respectively, show that
   $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$
   and each $= \frac{1}{2abc}\sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4}$.

8. If in the triangle ABC, C is a right angle, show that
   $$\tan(A \sim B) = \frac{a^2 \sim b^2}{2ab}.$$

9. Find an expression for the radius of the circle inscribed in a triangle.

   The points where the circle inscribed in a triangle ABC touches the sides are joined. Show that the sides of the triangle thus formed are proportional to $\cos\frac{A}{2}$, $\cos\frac{B}{2}$, $\cos\frac{C}{2}$ respectively.

10. In the triangle ABC, $b = 5$, $c = 4$, $A = 60°$. Find B and C, having given

$$\log 3 = \cdot 477121.$$
$$\text{L tan } 10° 53' = 9 \cdot 283907.$$
$$\text{L tan } 10° 54' = 9 \cdot 284525.$$

11. Two circles, radii a and b, touch one another externally. Show that the angle between their common tangents is

$$2 \tan^{-1} \frac{1}{2} \left(\sqrt{\frac{a}{b}} \sim \sqrt{\frac{b}{a}} \right).$$

PURE MATHEMATICS. (1.)

1. When is one plane perpendicular to another plane? What is the inclination of a plane to a plane?

   Find the inclination of the diagonal of a cube to any one of its faces.

2. Draw a straight line perpendicular to a given plane from a given point outside of that plane.

   Draw a plane which shall pass through a given straight line and shall be perpendicular to a given plane.

3. What is meant by the pole of a circle described on a sphere?

   If on a given sphere there be a small circle, find its radius in terms of the radius of the sphere and of the distance measured along the surface of the sphere of the small circle from the pole.

4. Prove that if a given right circular cone be cut by a plane which cuts all the generating lines, and if two spheres be inscribed in the cone touching the plane in two points S and S', then, if P be a point on the curve in which the cone is cut by the plane, SP + S'P is constant.

5. Prove that the area of a circle is given by the expression πr^2, where r is the radius of the circle and π is twice the circular measure of a right angle.

6. Assuming that

$$(\cos\theta + \sqrt{-1}\sin\theta)^p = \cos p\theta + \sqrt{-1}\sin p\theta$$

   when p is a positive integer, prove that the same formula holds when p is negative, and also when p is fractional.

   In the case where $p = \frac{1}{n}$ show how to get the rest of the n^{th} roots of $\cos\theta + \sqrt{-1}\sin\theta$.

7. Find the co-ordinates of the point of intersection of the straight lines

$$Ax + By + C = 0$$
$$A'x + B'y + C' = 0.$$

   Find the condition that the straight line $ax + by + c = 0$ should pass through the same point.

8. Find the length of the perpendicular from the point h, k on the straight line $ax + by + c = 0$.

   Find the equation to this perpendicular.

9. Find the radius and the co-ordinates of the centre of the circle whose equation is

$$10x - 6y - 9 = 2x^2 + 2y^2.$$

   Express this equation in polar co-ordinates with the same origin and the line passing through the centre as initial line. Also express the equation to the same circle when referred to rectangular axes whose point of intersection is on the circumference and which are equally distant from the centre.

10. Find the equation to the tangent at the point $x'y'$ to the circle

$$(x - a)^2 + (y - b)^2 = c^2.$$

Find the equation to the locus of the point of intersection of the tangent to the circle $x^2 + y^2 - 2ax - 2by = 0$, and the perpendicular to it drawn from the origin.

11. Given that the equation to the tangent to the parabola $y^2 = 4ax$ at the point $x'y'$ is

$$yy' = 2a(x + x'),$$

find the equation to the normal at that point, and prove that it may be put in the form—

$$y = mx - 2am - am^3.$$

12. Show that if p be the length of the perpendicular from the focus on the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $x'y'$, then

$$p^2 = \frac{b^2(a - ex')}{(a + ex')}.$$

Hence prove that the tangent is equally inclined to the focal radii.

13. Define a diameter of an ellipse.

Prove that the locus of middle points of a series of parallel chords of an ellipse is a straight line passing through the centre.

PURE MATHEMATICS. (2.)

1. Show how straight lines may be drawn bearing to a given straight line the ratios respectively of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$.

2. Find that point in a given straight line at which two given points subtend the greatest angle.

3. Through a given point within a circle a chord is drawn, and from O, the centre of the circle, OP is drawn perpendicular to the chord, and produced to Q, so that $OP \cdot OQ = OC^2$, where OC is a radius of the circle. Prove that the locus of Q is a straight line.

4. Solve the equation
$$\frac{x}{b} = \frac{x}{a + \cfrac{x}{a + \text{ad inf.}}}$$

5. Prove that approximately, when x is very small,
$$\frac{3\left(x + \frac{4}{9}\right)^{\frac{1}{2}}\left(1 - \frac{3}{4}x^2\right)^{\frac{1}{2}}}{2\left(1 + \frac{9}{16}x\right)^2} = 1 - \frac{307}{256}x^2.$$

6. How many different throws can be made with a pair of ordinary dice?

   If the faces of one of the dice were marked from 1 to 6 and those of the other from 5 to 10, how many different throws could be made?

7. If $m^x + n^y = a$, and $m^{x+2} + n^{y+2} = b$, find x and y.

8. If $\tan\theta + \tan 2\theta + \tan 3\theta = 0$, find θ.

9. The centres of the escribed circles of a given triangle are joined; find the area of the triangle thus formed in terms of the sides of the given triangle.

10. Eliminate α and β from the equations
$$x = (a \sin^2\alpha + b\cos^2\alpha)\cos^2\beta + e\sin^2\beta.$$
$$y = a\cos^2\alpha + b\sin^2\alpha.$$
$$z = (b - a)\sin\alpha\cos\alpha\cos\beta.$$

11. Show that sin 9° lies between ·156 and ·157.

12. Find the ratios $x : y : z$ from the equations

$$x^2 (y - z) = a^3.$$
$$y^2 (z - x) = b^3.$$
$$z^2 (x - y) = c^3.$$

PURE MATHEMATICS. (3.)

1. If $a_1\ a_2\ a_3 \ldots a_n$ are the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0,$$

show that $p_1^2 - 2p_2 = a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2$;

and $$\frac{p_{n-1}}{p_n} = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n}.$$

2. Transform the equation $p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$ into another in which the roots are the reciprocals of the roots of the given equation.

3. Solve the equations

(1.) $x^3 - 9x^2 + 23x - 15 = 0$, in which one root is half the sum of the other two.

(2.) $x^5 - x^3 + 4x^2 - 3x + 2 = 0$, which has equal roots.

4. Explain the meaning of "limit": find the limit of $(1 + ax)^{\frac{1}{x}}$ when $x = 0$; and differentiate from first principles $u = \log_a x$.

5. Differentiate $x^{\tan^{-1}x}$, and find the r^{th} differential coefficient of $e^x x^n$.

6. Show how fractions may be evaluated which assume the forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ for particular values of the variable.

Evaluate $\dfrac{\sin x \tan x}{1 - \cos x \cos 2x}$ when $x = 0$.

7. Change the expression

$$\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}}$$ into its equivalent in terms of r and θ, where $x = r\cos\theta$, $y = r\sin\theta$, and θ is the independent variable.

8. How do you detect the greatest and least values of functions of one variable?

A circle is described with its centre on the major axis of an ellipse, its semi-axes being a and b, at a distance $\frac{a^2 - b^2}{2a}$ from the centre of the ellipse, and with radius $\frac{a^2 + b^2}{2a}$. Show that the greatest chord of the circle which is also a tangent to the ellipse, is equal to half the distance between the foci.

9. Prove Maclaurin's theorem, and by it expand $y = \log_e (1 + x)$.

10. If r, θ be the polar co-ordinates of any point on a plane curve, and p the perpendicular from the pole upon the tangent at that point, prove that the radius of curvature at the point is equal to $r\frac{dr}{dp}$.

11. Explain what is meant by the evolute of a plane curve; and prove that the normal to the curve at any point is the tangent to the evolute.

The evolute of the spiral $p = kr$ is a similar spiral.

12. Integrate the expressions

$$\sqrt{a^2 - x^2}\,dx\,;\quad e^{ax}\cos ax\,dx.$$

13. Find the area included between the curves

$$y^2 = 4ax \text{ and } x^3 - y^3 = a^3.$$

14. Reduce the form $\int \sin^m x \cos^n x dx$ so as to be integrable; and show that

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cos^4 x dx = \frac{3.1.2}{7.5.3} \int_0^{\frac{\pi}{2}} \sin x dx.$$

STATICS.

1. Explain the propriety and advantage of representing statical forces by straight lines. How are statical forces measured numerically? Explain the application of this measure to the forces "pressure," "tension," "resistance," "friction."

   Define density. A cubic foot of a substance weighs 8 cwt., what volume of another substance five times as dense will weigh 14 cwt.?

2. Define the resultant of any number of statical forces. If three commensurable forces act upon a particle in one plane, prove generally the condition that must hold, so far as their directions are concerned, in order that these forces may keep the particle at rest; if two of the forces are equal, find the magnitude and direction of the third force that will keep the particle at rest.

   Example: Equal forces of 10 pounds acting at an angle of 120°.

3. Find a point within a triangle from which if straight lines be drawn to the angles of the triangle, the forces represented by those lines shall be in equilibrium.

   Show that the moment of any two of those forces round a point without the triangle in the same plane will be equal to the moment of the third force round the same point.

4. If any number of forces act in one plane on a rigid body, state and prove the conditions of equilibrium, and show that if they are not in equilibrium they must be equivalent to a single resultant or to a single couple.

5. When a body is placed with its base on a horizontal plane, examine the condition of its standing or falling: if the condition is fulfilled theoretically, is the stability of the body always practically secured?

   A number of bricks, each 9 inches long, 4 inches wide, and 3 inches thick, are placed one upon another in such a way that, whilst their narrowest surfaces (or thicknesses) are in the same vertical plane, each brick overlaps the other by half an inch of its length, the lowest brick resting on a horizontal plane; how many bricks may be so piled without falling?

6. What is meant by mechanical advantage being gained or lost in the application of a lever? Give two examples in reference to contrivances in common use where mechanical advantage is lost by a lever.

   Two weights of 8 ounces and 4 ounces are in equilibrium at the opposite ends of a straight lever without weight; if 2 ounces be added to the greater weight, the fulcrum must be moved through $\frac{4}{7}$ of an inch for equilibrium; find the length of the lever.

7. When a body is placed on a rough inclined plane, explain how the direction of the resistance of the plane is modified by friction when the body is in a state bordering upon motion.

A weight is placed on a rough inclined plane, and is just on the point of being moved up the plane by a force (P) parallel to the plane; find the force P when the angle of the inclined plane and the limited angle of resistance are given.

A plane of small slope rises one foot vertical for (n) feet horizontal, and the coefficient of friction is $\frac{1}{m}$. If a weight W be placed on the plane, show that the force P which will just move W up the plane is nearly $W\left(\frac{1}{m}+\frac{1}{n}\right)$.

8. Describe the construction of the screw in its simplest form, considered as a mechanical power.

Show that in the case of a rough screw, when the friction acts upon the screw, the equation for the state bordering on motion is

$$Pa = Wr \tan (\alpha + \phi),$$

when (a) is the length of the arm at which P acts, (r) the radius of the screw, (α) the angle of inclination of the thread, (ϕ) the limiting angle of resistance, W the weight placed at the end of the screw, which the screw supports.

9. How is work measured? Explain the term horse-power; state and explain how it is measured. Distinguish between the number of units of work applied to the machine and the number of units of useful work.

The diameter of the circular piston of a steam-engine is 60 inches, it makes 11 strokes per minute, the length of each stroke is 8 feet, the mean pressure per square inch on the piston is 15 lbs., the modulus of the steam-engine being ·65. Determine the number of cubic feet of water it will raise per hour from a well the depth of which is 300 feet, a cubic foot of water weighing 2·5 lbs.

10. Find the condition of stability of a heavy body, whose under surface is curved, resting on a horizontal plane, friction being sufficient to prevent sliding.

A hemisphere and rigid cylinder of the same material, having a common base, are cemented together; find the height of the cylinder so that when the hemisphere is placed with its convex surface on a horizontal plane, the equilibrium may be neutral, it being given that the centre of gravity of a hemisphere is at a distance from the centre of the sphere equal to ($\frac{3}{8}$ths) of the radius.

DYNAMICS.

1. Define velocity and acceleration.

   If f denote an acceleration when one foot and one second are taken as units, find an expression for the same acceleration when the units are m feet and n seconds.

2. Establish the formulæ $s = vt + \frac{1}{2}ft^2$, $v^2 = 2fs$.

   The space described by a body, falling from rest, in the last second of its motion is to that described in the last but one as 3 to 2; find the height from which the body was dropped, and the velocity with which it strikes the ground.

3. Prove that the time of falling from rest down a chord of a vertical circle drawn from the highest point is constant.

   AB is a quadrant of a circle whose centre is O, the radius OB being horizontal, C is a point on the quadrant, and the angle $BOC = \theta$. Show that the time in falling from A to C is to that in falling from C to B as $\sqrt{\text{Cos}\,\frac{\theta}{2}}$ to $\sqrt{\text{Sin}\,\frac{\theta}{2}}$.

4. State the laws of Motion.

5. Prove that the path of a projectile in vacuo is a parabola, and find an expression for its latus rectum. Show how to determine the position of the focus of the parabola.

6. A particle is projected with velocity V at an angle α to the horizon. Find expressions for the time (1) from the time of projection till the particle strikes a plane passing through the point of projection, and inclined at an angle β to the horizon, (2) from the time of projection till the particle is moving parallel to the inclined plane.

7. Two heavy bodies, the mass of one of which is double that of the other, are connected by a string which passes over a fixed smooth pulley. Show that the heavier body descends with an acceleration equal to $\frac{1}{3}$ of that with which it would move if it fell freely.

8. One body impinges directly on another. Show how to determine their velocities after impact, the elasticity being imperfect.

   A series of n perfectly elastic balls stand in a straight line, the mass of each ball being double that of the one preceding. The first ball is projected towards the second with velocity V, the second ball then impinges on the third, and so on. Find the velocity of the n^{th} ball.

9. Show that the velocity acquired by a particle descending from rest to the lowest point of a smooth tube in the form of a vertical circle, varies as the chord of the arc of descent.

   A particle slides from rest from the highest point of a sphere. Find where, and with what velocity, it will leave the sphere.

10. Find in what direction a ball must be projected along a smooth horizontal table from a given point, so that after reflexion at a given fixed smooth vertical plane it may pass through another given point; the index of elasticity being supposed known.

11. A string passing through a small hole in a smooth horizontal table has a small sphere of mass m attached to each end of it; the upper sphere revolves in a circle on the table, when suddenly it strikes an obstacle and loses half its velocity: find what diminution must be made in the weight of the lower sphere in order that the upper sphere may continue revolving in a circle.

12. A particle of given elasticity slides down a smooth plane inclined at the angle α to the horizon, and impinges on a fixed smooth horizontal plane at the foot of the former. Find the range of the particle after rebounding from the latter plane.

13. If a pendulum loses n seconds a day, find approximately what alteration should be made in its length.

SET VIII.

EUCLID.

1. Explain what is meant by an "Axiom." Give any axiom about parallel straight lines, and mention a theorem which is founded upon it.

2. If two angles of a triangle be equal to each other, the sides also which subtend the equal angles shall be equal to one another.

3. If a side of a triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.

   ABC is an isosceles triangle having the sides AB and AC equal. CD is drawn at right angles to AB. Show that the difference of the angles DCA, DCB is equal to the difference of the angles A, B.

4. In any right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the sides.

   ABCD is a quadrilateral having the diagonals AC, BD at right angles. Show that the squares on AB, CD are together equal to the squares on AD, BC.

5. In obtuse-angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.

6. Show how to draw a straight line from a given point, either without or in the circumference, which shall touch a given circle.

7. If from any point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square on the line which touches it.

   In the base AB of a triangle ABD a point C is taken such that the rectangle AB, AC is equal to the square on AD. Show that the angle A is equal to the difference of the angles DBC, DCB.

8. In a given circle, show how to inscribe a triangle equiangular to a given triangle.

Show how to inscribe a circle in a given rhombus.

9. Similar triangles are to one another in the duplicate ratio of their homologous sides.

The sides of a regular hexagon ABCDEF are produced both ways, forming with each other six points of intersection A′, B′, C′, D′, E′, F′. Show that the hexagon A′B′C′D′E′F′ is three times as great as ABCDEF.

10. Equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides.

Two circles touch one another externally at A. A straight line touches the circles at B and C, and is produced to meet the straight line joining their centres at T. Show that the triangles TCA, TAB are similar.

ALGEBRA.

1. Multiply $(1 + 2x^{\frac{1}{2}} + 3x + 4x^{\frac{3}{2}})$ by
$(1 - 2x^{\frac{1}{2}} + 3x - 4x^{\frac{3}{2}})$.

2. Divide $x^4 - 2bx^3 - (a^2 - b^2)\,x^2 + 2a^2bx - a^2b^2$ by
$x^2 + (a - b)\,x - ab$.

3. Prove
$$\frac{(x-a)\,(x-b)}{(x-c)\,(x-d)} = 1, \text{ when } \frac{ab - cd}{(a+b) - (c+d)}$$
is substituted for (x).

4. *State* the rule for finding the least common multiple of two algebraical expressions (A) and (B). When will the product of (A) and (B) be their least common multiple?

If (M) be the least common multiple of (A) and (B), prove that any other common multiple of (A) and (B) is a multiple of (M).

Find the least common multiple of $(2x - 1)$, $(6x^2 - x - 1)$, and $(8x^3 - 10x + 3)$; and express it as the product of binomial factors.

5. Solve the following equations:—

(1.) $\sqrt{x} + \sqrt{x - 64} = \dfrac{96}{\sqrt{x - 64}}$.

(2.) $\dfrac{2x}{x-4} - \dfrac{2x-5}{x-3} = 3\frac{2}{3}$.

(3.) $\begin{cases} x\,(y + z - x) = 39 - 2x^2. \\ y\,(x + z - y) = 52 - 2y^2. \\ z\,(x + y - z) = 78 - 2z^2. \end{cases}$

6. Divide $(5^{\frac{1}{2}} + 9^{\frac{1}{2}})$ by $(5^{\frac{1}{4}} + 9^{\frac{1}{4}})$, and extract the square root of the quotient in the form of a binomial surd.

7. Substitute $(y + 1)$ for (x) in the equation
$$x^3 - 2x^2 - 5x + 6 = 0,$$
and solve the equation that results from the substitution. Without solving the equation
$$6x^2 - 96x + 378 = 0,$$
find the sum of its roots, the difference of its roots, and the sum of the squares of its roots.

8. A and B are two stations 300 miles apart. At the same moment locomotives which travel uniformly start from A and B, each to the opposite station. The locomotive from A reaches B, 9 hours, the locomotive from B reaches A, 4 hours, after they pass each other on the road: find the rate at which each locomotive travels, and the time in which either of them performs the whole distance.

9. Show how to transfer a decimal fraction from the common scale of notation to any other scale whose radix is (r); if the given decimal be finite, will the transformed decimal necessarily be so?

   Find the radix of the scale in which $124 \cdot 96$ in the denary scale is expressed by $444 \cdot 44$.

10. When a, b, c are three numbers differing in magnitude, show that if they are in arithmetic they cannot be also in geometric progression. Insert (m) geometric means between $\frac{b}{a}$ and $\frac{a}{b}$, and show that whether (m) be even or odd, the product of all the terms of the series will be unity.

11. The first term of an arithmetic series is $n^2 - n + 1$, and the sum of (n) terms is n^3; find the common difference. Hence show that the cube of every integer (n) is the sum of (n) consecutive odd numbers. Of what such odd numbers is 64 the sum?

12. If $C(n)_r$ denote the number of combinations of (n) things taken (r) together, find $C(n)_r$, assuming the formula for the number of permutations of (n) things taken (r) together.

    Adopting the above notation, prove

    $$C(n)_r = C(n-1)_r + C(n-1)_{r-1},$$

    and verify the last equation when $n = 7$ and $r = 4$.

13. Assuming the expansion of $(1+x)^n$ by the binomial theorem when the number of terms is finite, show that the coefficients of the r^{th} terms from the beginning and end of the series respectively are the same.

    Show how to find the greatest term of $(1+x)^n$, and examine the conditions when there may be two terms equal to each other, and each greater than any of the other terms. In the expansion of $\left(1+\frac{2}{3}\right)^9$ prove that there are two such terms.

TRIGONOMETRY.

1. Show how to transform the circular measure of an angle into degrees.

   Find the unit, when the circular measure of 120° is represented by $\frac{15}{16}$ [$\pi = 3{\cdot}1416$].

2. Show that $\sin(90° + A) = \cos A = \sin(90° - A)$; and find the value of the trigonometrical functions for an angle of 2670°.

3. Prove that

   $\mathrm{Cos}(A + B) = \cos A \cos B - \sin A \sin B$ when A is a positive angle between $\frac{\pi}{4}$ and $\frac{\pi}{2}$, and B a negative angle between $-\pi$ and $-\frac{5\pi}{4}$.

   Find the values of $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ in terms of $\cos A$.

4. Obtain the values of $\sin 18°$ and $\cos 75°$. Show that

$$\frac{1 + \cot 60°}{1 - \cot 60°} = \left\{\frac{1 + \cos 30°}{1 - \cos 30°}\right\}^{\frac{1}{2}}.$$

5. Prove the following identities:—

   i. $\mathrm{Cos}\, 3A = 4\cos^3 A - 3\cos A$.

   ii. $\mathrm{Tan}^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4} = \sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1} 3$.

6. In any triangle ABC

   i. $\mathrm{Sin}\frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}$.

   ii. $\mathrm{Tan}\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$.

7. ABCD is a quadrilateral whose opposite sides, AB and CD, meet at right angles: show that

   $\cot \mathrm{DAB} \cot \mathrm{ABC} = \cot \mathrm{CAB} \cot \mathrm{ABD}$.

8. If $r\, r_a\, r_b\, r_c$ are the radii of the circle inscribed in, and of the circles escribed to the triangle ABC, then

$$r_a \cot\frac{A}{2} = r_b \cot\frac{B}{2} = r_c \cot\frac{C}{2} = r \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

9. Explain geometrically the meaning of the ambiguous case in the solution of triangles.

If a b are the two given sides, $a < b$, A the given angle, and C C′ the two values of the ambiguous angle, then

$$\cos A \cos B = \sin \frac{C + C'}{2} \sin \frac{C - C'}{2}.$$

10. Show that in general the increment of the tabular logarithmic cosine of an angle is approximately proportional to the increment of the angle; and being given

$$\text{L} \cos 55° 11' = 9{\cdot}7565999,$$
$$\text{L} \cos 55° 12' = 9{\cdot}7564182,$$

find the tabular logarithm of the cosine of 55° 11′ 12″.

11. The angular elevation of a tower at a place due south of it is 45° and at another place due west of the former the elevation is 15°. Show that the height of the tower is

$$\frac{a}{2}[3^{\frac{1}{2}} - 3^{-\frac{1}{2}}],$$

where a is the distance between the two places.

PURE MATHEMATICS. (1.)

1. If a solid angle be contained by three plane angles, any two of them are greater than the third.

2. If two parallel planes be cut by another plane, their common sections with it are parallel.

   If three straight lines OA, OB, OC, are cut by parallel planes in ABC, A'B'C', show that the triangles ABC, A'B'C' are similar.

3. If a straight line be at right angles to a plane, every plane which passes through it shall be at right angles to that plane.

   When is it possible to draw through a given straight line a plane perpendicular to another given straight line?

4. Define a sphere; and prove that the line of intersection of two spheres is a circle.

5. Prove that the curve of intersection of a right cone, made by a plane which is parallel to one of the generating lines, is a parabola.

   Show that the distance of the vertex of the parabola from its focus is in a constant ratio to its distance from the vertex of the cone.

6. Prove that the circumferences of circles vary as their radii. For what purpose does Trigonometry avail itself of this property of circles?

7. Show that $\cos\frac{p}{q}\theta + \sqrt{-1}\sin\frac{p}{q}\theta$ is one of the values of

   $$[\cos\theta + \sqrt{-1}\sin\theta]^n, \text{ where } n = \frac{p}{q}.$$

   Hence find the values of $(-1)^{\frac{1}{3}}$.

8. Find the sum of the series

   $$\sin a + \sin 2a + \sin 3a + \ldots \text{ to } n \text{ terms.}$$

9. Find an expression for the tangent of the angle contained between the two straight lines

   $$ax + by + p = 0 \text{ and } x\cos a + y\sin a - p = 0.$$

   If they enclose an angle $\frac{\pi}{4}$ between them and meet the straight line $x\sin a - y\cos a = 0$ in the same point, show that $a^2 + b^2 = 2$.

10. What is the area of the triangle, the equations to whose sides are

$$a_1x + b_1y = 0, \quad a_2x + b_2y = 0, \quad a_3x + b_3y = c?$$

11. What is represented by

$$(x - a)^2 + (y - b)^2 - r^2$$

when the point (x, y) is (1) outside, (2) inside, (3) on the circle, the co-ordinates of whose centre are (a, b) and whose radius is r?

12. Find the polar equation of the circle in its most general form.

Prove that when the pole is on the circumference and the diameter is taken as the initial line, the equation of a tangent becomes $r \cos(\theta - 2\alpha) = 2c \cos^2\alpha$, where α is the angular co-ordinate of the point of contact, and c the radius of the circle.

13. The equation of a parabola being $y^2 = 4ax$, and the equation of a chord being $y - x\sqrt{2} + 4a\sqrt{2} = 0$, prove that the chord is a normal to the curve, and that its length is $6\sqrt{3}a$.

14. Prove that the portion of the tangent to an hyperbola, which the asymptotes intercept, is bisected at the point of contact.

PURE MATHEMATICS. (2.)

1. By the application of algebra to geometry divide a straight line into two parts, so that the rectangle contained by the whole line and one of the parts shall be equal to the square of the other part.

   If the solution is obtained by a quadratic equation and one root does not apply to the question as proposed, state the geometrical problem to which that root will apply.

2. If P be a point without two concentric circles, whose common centre is A, PM, PN tangents to the circles in M and N, C the middle point of AP, prove CM = CN.

3. Two circles whose centres are O and Q touch in C, BE, a common tangent to both circles in B and E meets QO produced in A, a common tangent at C intersects BE in D:

   Prove (1) ODQ a right angle.

   (2) BE a mean proportional to the diameters of the circles.

4. To complete a certain work, a workman A alone would take (m) times as many days as B and C working together, B alone would take (n) times as many days as A and C together, C alone would take (p) times as many days as A and B together:

   Show that the number of days in which each would do it alone are as $m+1 : n+1 : p+1$. Prove also

$$\frac{m}{m+1} + \frac{n}{n+1} + \frac{p}{p+1} = 2.$$

5. A cottage at the beginning of a year was worth £250, but it was found that by dilapidations at the end of each year it lost ten per cent. of the value it had at the beginning of each year; after what number of years would the value of the cottage be reduced below £25?

   Given $\log_{10} 3 = \cdot 4471213$.

6. If $\frac{a}{a+b}$ represent the probability of an event happening in one trial, what are (a) and (b)?

   What probabilities will be represented by

$$\left(\frac{a}{a+b}\right), \quad \frac{na^{n-1}b}{(a+b)^n}, \quad \frac{n(n-1)}{1.2}\frac{a^{n-2}b^2}{(a+b)^n},$$

   respectively, (a), (b), and (n) being whole numbers?

   A bag contains 3 white and 7 black balls, what is the probability of drawing one white ball and two black balls in three trials, drawing one ball at a time, the balls being replaced after each drawing? What is the probability if the balls are not replaced?

7. Prove the infinite series—

$$1 + \frac{3}{2^3} + \frac{1.3}{1.2} \cdot \frac{3^2}{2^6} + \frac{1.3.5}{1.2.3} \cdot \frac{3^3}{2^9} + \&c. = 2.$$

8. Solve the equation $5^x + 5^{-x} = 125 \cdot 008$.
   Given, if required, $\log_{10} 2 = \cdot 3010300$.

9. If a, b, c, the sides of a triangle, be in arithmetical progression, prove $2 \sin \frac{A}{2} \cdot \sin \frac{C}{2} = \sin \frac{B}{2}$, A, B, C being the angles subtended by the sides a, b, c, respectively.

10. Prove $\tan \frac{15^\circ}{2} = (\sqrt{3} - \sqrt{2})(\sqrt{2} - 1)$.

11. Given $\cos(\alpha - \beta) \cos \alpha = a \sin \beta \cos \beta$

$$\sin(\alpha - \beta) \cos \alpha = \frac{a}{2} \cos^2 \beta,$$

Prove $a = \dfrac{2 \sin \beta}{1 + 3 \sin^2 \beta}$.

PURE MATHEMATICS. (3.)

1. Write down the quotient and remainder arising from the division of $x^n + p_1 x^{n-1} + p_2 x^{n-2} +$ &c. by $x - a$. Apply the principle in the case where $3x^3 + 17x^2 + 10x - 14$ is divided by $x - 4$.

2. Prove that in an equation with real coefficients imaginary roots occur in pairs. One of the roots of the equation $x^4 + 4x^3 + 5x^2 + 2x - 2 = 0$ is $-1 + \sqrt{-1}$; find the other roots.

3. Find the equation each of whose roots differs by the same quantity from a root of a given equation; and, as an example, find the equation each of whose roots is greater by unity than a root of the equation $x^3 - 5x^2 + 6x - 3 = 0$.

4. Determine the limit of $(1 + nx)^{\frac{1}{n}}$, and of $\frac{\log(1+x)}{x}$, when x has zero for its limit; and find the differential coefficients of

$$\tan^{-1}\left[x(1+3x)\right] \text{ and } e^{2nx}\left(\frac{1-nx}{1+nx}\right)^{\frac{1}{2}}.$$

5. Expand $\tan^{-1}e^x$ to four terms by Maclaurin's Theorem.

6. Proves Leibnitz's Theorem—

$$\frac{d^n(u.v)}{dx^n} = u.\frac{d^nv}{dx^n} + n\frac{du}{dx}.\frac{d^{n-1}v}{dx^{n-1}} + \frac{n.n-1}{1.2}\frac{d^2u}{dx^2}.\frac{d^{n-2}v}{dx^{n-2}} + \text{ &c., and}$$

apply it to find the general differential coefficient of $x^m \sin mx$.

7. Investigate a method for determining the maxima and minima values of a function of one variable.

PSp is a chord passing through the focus S of an ellipse, determine the position of the chord so that the sum of the perpendiculars from P and p on the major axis may be a maximum.

8. Prove that a curve is convex or concave to the axis of x according as y and $\frac{d^2y}{dx^2}$ have the same or different signs.

9. Find an expression for the radius of curvature, and apply it to find the curvature at the point where the curve $\left(\frac{x}{a}\right) + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1$ meets the axis of x.

10. In curves referred to polar co-ordinates, find an expression for the subtangent: and show that the subtangent of $r\theta = a$ is a constant.

11. Define a surface of revolution ; and find an expression for the rate of increase of the portion of its superficial area cut off by a plane which moves along the axis and is perpendicular to it.

12. Integrate the expressions

(i.) $\dfrac{dx}{a^2 + x^2}$.

(ii.) $\dfrac{(2x + 1)\,dx}{(x - 1)^2 (x^2 + 1)}$.

(iii.) $e^{ax} \cos nx\, dx$.

13. Explain the term "definite integral."

Find the value of $\displaystyle\int_0^{\frac{\pi}{2}} (\cos\theta)^{2n}\, d\theta$.

14. Trace the curve $y^3 = (x + 2)^2 (x + 4)$, and find the whole area of the curve $a^4y^2 + b^2x^4 = a^2b^2x^2$.

STATICS.

1. If three forces acting at a point are in equilibrium, show that each force is proportional to the sine of the angle between the other two.

   A circular lamina of mass M rests in a vertical position on two smooth pegs in the same vertical plane. Find the pressures on the pegs, their positions being at angular distances θ, φ respectively from the lowest point of the lamina.

2. Find an expression for R the resultant of two forces P, Q acting at a point and inclined at an angle θ.

   Two forces acting at an angle of 60° have a resultant $2\sqrt{3}$ lbs. If one of the forces is 2 lbs., find the other force.

3. When a body can only turn round a fixed axis and is acted on by a system of forces in a plane perpendicular to the axis, such that the algebraical sum of their moments round the axis vanishes, show that the body will be in equilibrium.

   A lamina of weight W in the shape of a rhombus is suspended from one of its angular points, and hangs freely. Find what weight must be attached to the extremity of one of the sides which meet in the point of suspension, in order that the other may be horizontal.

4. Define the centre of gravity of (1) two or more particles, (2) a solid body.

   A number of particles of masses m_1, m_2, m_3, &c., are situated in the same plane; show how to find the position of their centre of gravity.

   ABCD is a square lamina; if the circle inscribed in the triangle ABC is cut out, find the centre of gravity of the remaining area.

5. A straight lever is acted on by forces in the ratio $\sqrt{3}+1 : \sqrt{3}-1$ at its extremities, and inclined at angles 60° and 30° to its length. Find the magnitude of the pressure on the fulcrum, and the direction in which it acts.

6. Explain what is meant by a "couple." Prove that a single force and a couple in one plane are equal to the same single force acting in a direction parallel to its original direction. How are couples compounded?

7. Enunciate the "principle of virtual velocities." Prove the truth of the principle in the case of the wheel and axle.

8. In the system of pulleys in which each pulley hangs by a separate string, all the strings being parallel and the power acting upwards, find the relation between the power and the weight when the system is in equilibrium.

If there are three pulleys it is found that a certain weight can be supported by a power of 7 lbs., but if there are four pulleys the same weight can be supported by a power of 4 lbs.; find the weight supported, and the weight of the pulleys: the pulleys being of equal weight.

9. A square table stands on four legs, placed respectively at the middle points of its sides; find the greatest weight that can be placed at one of the corners without upsetting the table.

10. State the laws of Friction.

A body is supported on a rough inclined plane by a force acting along the plane. If the greatest magnitude of the force when the plane is inclined at the angle α to the horizon is equal to the least value of the force when the plane is inclined at the angle α' to the horizon, show that

$$\mu = \frac{\sin \alpha \sim \sin \alpha'}{\cos \alpha + \cos \alpha'}$$

μ being the coefficient of friction.

DYNAMICS.

1. Explain how velocity is measured (1) when constant, (2) when variable.

   A body, moving uniformly, passes over a mile in fifteen minutes; if 64 be the measure of the velocity, and eleven feet be the unit of length, find the unit of time.

2. State and prove the parallelogram of velocities.

   A man is riding, at the rate of 12 miles an hour, along a straight road at right angles to a railway, on which a train is passing at the rate of 16 miles an hour; find the velocity of the man relative to the train.

3. Explain how acceleration is measured.

   What is the measure of the acceleration due to gravity (1) when a foot and half a second are units of length and time, (2) when the units are a mile and eleven seconds?

4. State the Second Law of Motion.

   A ball, which is at rest on a smooth horizontal table, receives, at the same instant, two impulses in given directions; employ the Second Law of Motion and the parallelogram of velocities to determine the motion produced, the velocities due to each impulse being known.

5. A ball is projected vertically upwards with a velocity of 128 feet per second; find after what time it will have attained a height of 192 feet above the point of projection, and explain the double result.

   Find also the greatest height attained.

6. Two weights, P and Q, are connected by an inextensible string which passes over a fixed smooth pulley; determine the acceleration of the system when in motion under the action of gravity only, and find the tension of the string.

   If P = 2 Q, and if the system have no motion initially, find the velocity at the end of three seconds, and determine the instantaneous effect and the subsequent change of motion produced by suddenly attaching at that time a weight Q to the ascending body.

7. A body is projected with a velocity u in a direction inclined at the angle α to the vertical; if at the time t the direction of its motion be inclined at the angle θ to the vertical, prove that

$$\tan\theta = \frac{u \sin\alpha}{u\cos\alpha - gt},$$

   and that its velocity $= \sqrt{u^2 + g^2t^2 - 2ugt\cos\alpha}$.

   At what time will the body be moving at right angles to the direction of projection?

8. Find the range of a projectile on the horizontal plane through the point of projection.

   If the velocity of projection be given, find the greatest range, and show that, for any range short of the greatest, there are two directions of projection, which are equally inclined to the direction of projection giving the greatest range.

9. Determine the change in the motion of a smooth inelastic ball produced by its impact on, 1st a fixed surface, 2nd a fixed edge.

   An inelastic ball, sliding along a smooth horizontal plane with the velocity of 16 feet per second, impinges upon a smooth horizontal rail at right angles to the direction of its motion; if the height of the rail above the plane be half the radius of the ball, prove that the latus rectum of the parabola subsequently described is one foot in length.

10. An elastic ball impinges in a given direction and with a given velocity upon a fixed smooth plane; the coefficient of elasticity being given, determine the velocity of the ball immediately after the impact.

    A ball of elasticity $\frac{1}{2}$ falls from a height of 64 feet upon a horizontal plane; find the height to which it will rise at the first rebound, and the time at which the rebounding will cease.

11. A shot weighing 700 lbs. is fired with a velocity of 1700 feet per second from a gun weighing 38 tons. Find the velocity with which the gun recoils, neglecting the weight of the powder.

    If the recoil be resisted by a constant pressure equal to the weight of 17 tons, through what space in feet will the gun recoil?

12. If a point move with uniform velocity v in a circle of radius r, prove that its acceleration is in the direction of the centre of the circle and is equal to $\frac{v^2}{r}$.

    Explain the difficulty of walking rapidly in a curve on smooth ice, and show how it is possible for a man, in skating, to describe a circle on one foot.

13. Write down the expression for the time of oscillation of a simple pendulum of length l, and find, approximately, the length of a pendulum which oscillates in one second.

SET IX.

EUCLID.

1. Define a plane superficies, an obtuse-angled triangle, a segment of a circle, and reciprocal rectilineal figures; and show how the latter differ from similar figures.

2. The angles at the base of an isosceles triangle are equal; and if the equal sides be produced, the angles at the other side of the base are equal.

3. Parallelograms on the same base, and between the same parallels, are equal to one another.

   ABCD is a parallelogram, C being the corner opposite to A. If P, Q are the middle points of CB and CD, show that the triangle APQ is equal to the sum of the triangles BPQ, CPQ, and DPQ.

4. In a right-angled triangle the square on the side subtending the right angle is equal to the sum of the squares described on the other two sides.

   If a quadrilateral be such that its diagonals are at right angles to one another, the sums of the squares described on the opposite sides are equal.

5. If a straight line be bisected, and produced to any point, the square on the whole line thus produced, and the square on the part produced, are together double of the square on half the line bisected, and of the square made up of the half and the part produced.

6. A segment of a circle being given, describe the circle of which it is a segment.

7. If two straight lines cut one another within a circle, the rectangle contained by the segments of one of them shall be equal to the rectangle contained by the segments of the other. Prove this for the case when one straight line only passes through the centre of the circle.

   ACDB is a semicircle, whose diameter is AB; AD, CB are any two chords meeting one another in P. Prove that

$$AB^2 = PA \cdot AD + PB \cdot BC.$$

8. When is a triangle said to be inscribed in a circle? In a given circle show how to inscribe a triangle equiangular to a given triangle.

Show that the perpendiculars drawn through the middle points of the sides of the triangle pass through the centre of the circumscribing circle.

9. Inscribe an equilateral and equiangular pentagon in a given circle.

If ABCDE be the inscribed pentagon, prove that

$$BE^2 = AB \cdot BE + AB^2.$$

10. The sides about the equal angles of equiangular triangles are proportionals; and those which are opposite to the equal angles are homologous sides, that is, are the antecedents or the consequents of the ratios.

O is the centre and OA the radius of a given circle, and V is the middle point of OA. P and Q are two points on the circumference on opposite sides of, and equally distant from A. QV is produced to meet the circle in L. Show that, whatever be the length of the arc PQ, LP always meets OA produced in a fixed point.

11. The rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle is equal to the sum of the rectangles contained by the opposite sides.

ALGEBRA.

1. Show that the result is the same whether $2 + \sqrt{3}$ or $2 - \sqrt{3}$ is substituted for x in the expression $x^3 - 15x + 5$, and find that result.

2. Prove
$$(x+y)^3 + 3(x+y)^2 z + 3(x+y) z^2 + z^3 = (x+z)^3 + 3(x+z)^2 y + 3(x+z) y + y^3.$$

3. Multiply
$$(x + y + 2\sqrt{x+y} + 4) \text{ by } (x + y - 2\sqrt{x+y} + 4).$$

4. Divide $(9a^2 + 6ab + b^2 - 4c^2 - 4cd - d^2)$ by $(3a + b - 2c - d)$.

5. Explain why, in finding the greatest common measure of two algebraical expressions, the successive remainders that appear in the process may be multiplied or divided by certain numbers or factors without affecting the greatest common measure.

   Find the greatest common measure of the numerator and denominator of the fraction
$$\frac{3x^2 - (4a + 2b)x + a^2 + 2ab}{x^3 - (2a + b)x^2 + (a^2 + 2ab)x - a^2 b}$$
   and reduce it to its lowest terms.

6. Solve the following equations:—

   (1) $\dfrac{(x-1)(x-2)(x-6)}{(x-3)^3} = 1.$

   (2) $\dfrac{2x+5}{2x-3} - \dfrac{3x-1}{x+1} = \dfrac{5}{3}.$

   (3) $\left\{\begin{array}{l} (x+1)^2 + (x+1)(y+2) + (y+2)^2 = 133 \\ (x+1) + \sqrt{(x+1)(y+2)} + (y+2) = 19 \end{array}\right\}$

7. A rifleman fires point blank at a target 500 yards distant from him, and hears the bullet strike the target $4\frac{1}{3}$ seconds after he fired; an observer standing at a distance of 400 yards from the target, and 650 yards from the shooter, hears the shot strike the target $2\frac{1}{3}$ seconds after he hears the report: find the velocity of sound and the velocity of the bullet, both velocities being supposed uniform.

8. Form the equation whose roots are $\dfrac{-1 \pm \sqrt{-3}}{2}$, and verify its correctness by afterwards solving it as a quadratic.

9. When are algebraical expressions called imaginary or impossible? Express the values of $(\sqrt{-1})^m$ when (m) has each of the four forms, $4n$, $4n+1$, $(4n+2)$, $(4n+3)$, (n) being a positive whole number.

Reduce $\frac{(a+b\sqrt{-1})^2}{a-b\sqrt{-1}} - \frac{(a-b\sqrt{-1})^2}{a+b\sqrt{-1}}$.

Show when $\sqrt{a+b\sqrt{-1}}$ may be expressed in a form similar to $p+q\sqrt{-1}$, and find in that form $\sqrt[3]{-64a^4}$.

10. If $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3} \ldots \frac{a_n}{b_n}$ be proper fractions arranged in order of ascending magnitude,

Prove

$$\frac{a_1+a_2+a_3+\&c.+a_n}{b_1+b_2+b_3+\&c.+b_n} > \frac{a_1}{b_1} < \frac{a_n}{b_n}.$$

If a, b, c express numerically the three sides of a plane triangle, prove

$$(a+b+c)^2 > 2\,(a^2+b^2+c^2).$$

11. Investigate an expression for (S) the sum of $(2n)$ terms of the arithmetic progression $a+(a-b)+(a-2b)$, &c. What is the least number of terms reckoned from (a) that will make S negative?

What does the expression for S become if $b = \frac{a}{n}$?

Find the number of terms of the series $24+21+18+$ &c. of which 78 is the sum, and explain the double answer.

12. Define Harmonic progression. Find the harmonic mean between a and c. If a, b, c are in harmonic progression, (x) the harmonic mean between (a) and (b), and (y) the harmonic mean between (b) and (c), prove that x, b, y are in harmonic progression.

13. Find the number of permutations of (n) things taken (r) together.

There are $(2m)$ things, (m) of which are alike of one kind and (m) of another; find the total number of permutations of the $(2m)$ things taken altogether, and show that the number is greater in this case than if there were $(m-r)$ things alike of one kind and $(m+r)$ alike of the other kind.

14. Assuming the form of the binomial theorem when the index is a positive whole number, prove it when the index is a positive fraction.

Express the 10th term of $(1+x)^{\frac{1}{2}}$ with a numerator containing the product of odd numbers and a denominator a power of 2.

Find by the binomial theorem $\sqrt[3]{1003}$ to five places of decimals.

TRIGONOMETRY.

1. Explain the English and French methods of measuring angles.
   Express according to the French method an angle containing $4\frac{1}{2}$ English minutes.

2. Define the *sine* of an angle, and find the greatest and least values which a sine can have.
   Prove that $\sin A = -\sin(180^\circ + A)$.

3. Show that
   $\cos(A - B) = \cos A \cos B + \sin A \sin B$.
   Express $2 \sin 5A \sin 9A$ as the difference of two cosines.

4. Find the tangents of 45°, 60°, and 15°.
   Write down a general formula for all angles whose tangent $= -1$.

5. Prove that
   (1.) $\sin 2A = \dfrac{2 \tan A}{1 + \tan^2 A}$.
   (2.) $\cot A + \operatorname{cosec} A = \cot \dfrac{A}{2}$.
   (3.) $\sin 18^\circ + \sin 30^\circ = \sin 54^\circ$.

6. Show that the circular measure of an angle less than a right angle is always intermediate in value between its sine and its tangent.
   Calculate roughly the distance at which a globe $5\frac{1}{2}$ inches in diameter will subtend an angle of 6′. $\left(\pi = \dfrac{22}{7}\right)$

7. One side of a triangle is double of another, and the included angle is a right angle.
   Find the other angles, having given
   $\log 2 = \cdot 30103$, L cot 26° 34′ $= 10 \cdot 3009994$,
   Diff. for 1′ $= \cdot 0003159$.

8. Prove that the sides of any triangle are proportional to the sines of the opposite angles.
   The altitude of a certain rock is observed to be 47°, and after walking towards it 1000 feet up a slope inclined at 32° to the horizon, the observer finds that the altitude is 77°. Find the vertical height of the rock above the first point of observation, having given $\sin 47^\circ = \cdot 73135$.

9. Find expressions for r, R, the radii of the inscribed and circumscribed circles of a triangle, and show that

$$r = 4\,\mathrm{R}.\sin\frac{\mathrm{A}}{2}.\sin\frac{\mathrm{B}}{2}.\sin\frac{\mathrm{C}}{2}.$$

10. Show that in any triangle—

$$(1.)\ \mathrm{Cos}\,\frac{\mathrm{A}}{2} = \sqrt{\frac{s\,(s-a)}{bc}}.$$

$$(2.)\ \mathrm{Tan}\,\frac{3\,\mathrm{A}}{2} = \frac{\sin 3\,\mathrm{B} - \sin 3\,\mathrm{C}}{\cos 3\,\mathrm{C} - \cos 3\,\mathrm{B}}.$$

11. The hypotenuse and shortest side of a right-angled triangle are 5 feet and 3 feet respectively. Find the length of the perpendicular from the right angle upon the hypotenuse, and show that it is inclined at an angle $\sin^{-1}\frac{7}{25}$ to the straight line drawn from the right angle to the middle point of the hypotenuse.

PURE MATHEMATICS. (1.)

1. Two straight lines which are each of them parallel to the same straight line, and not in the same plane with it, are parallel to one another.

   If A, B, C, and D be any four points in space, and if E, F, G, and H be the middle points of the lines AB, BC, CD, and DA, prove that EFGH is a parallelogram.

2. Draw a straight line perpendicular to a plane, from a given point above it.

   Prove that the angle between a straight line and its projection on any plane is less than the angle between the line and any other line meeting it in that plane.

3. If a solid angle be contained by three plane angles, the sum of these latter must be together less than four right angles.

4. Construct a sphere whose surface shall pass through four given points.

   In what cases does this become impossible?

5. Define the terms *right cone, axis, vertex, generating line.*

   Prove that the curve of intersection of a right cone with a plane is an ellipse, hyperbola, or parabola, except in certain special cases.

   If the vertical angle of the cone be 90°, find the inclination of the plane in order that the section may be an equilateral hyperbola.

6. Find cos 3A in terms of cos A.

   Prove that $2\sqrt{2}\cos\alpha$ is one root of the equation

   $$x^3 - 6x - 4\sqrt{2}\cos 3\alpha = 0.$$

7. Find $\cos^{2m}\theta$ in cosines of descending multiples of θ.

   Prove that $\cos^6\theta + \sin^6\theta = \frac{1}{8}(5 + 3\cos 4\theta)$.

8. Prove that

   $$\cos\theta + \sqrt{-1}\sin\theta = e^{\theta\sqrt{-1}}.$$

   Hence prove that

   $$\epsilon^{\theta}(\cos\theta + \sqrt{-1}\sin\theta) = \epsilon^{\theta\sqrt{2}}\left(\cos\tfrac{\pi}{4} + \sqrt{-1}\sin\tfrac{\pi}{4}\right).$$

9. Prove that the diagonals of the parallelograms formed by the four straight lines

   $$\sqrt{3}x + y = 0, \quad \sqrt{3}y + x = 0, \quad \sqrt{3}x + y = 1, \quad \sqrt{3}y + x = 1,$$

   are at right angles to one another.

10. Find the equation of the circle referred to any rectangular axes.

If the circle of radius a be referred to two diameters at right angles to one another, prove that the chord joining the points $x_1\ y_1$, $x_2\ y_2$ on the circumference has for its equation

$$y\,(y_1 + y_2) + x\,(x_1 + x_2) = y_1 y_2 + x_1 x_2 + a^2.$$

11. Prove either analytically or geometrically that two tangents can be drawn to a parabola from any external point P.

If the vertex of the parabola be A, and the points of intersection of the tangents with the axis be T_1 and T_2, prove that the distance of P from the tangent at A is a mean proportional between AT_1 and AT_2.

12. Prove that the straight line

$$y = mx + \sqrt{m^2 a^2 + b^2}$$

touches the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Hence prove that the sum of the squares of the perpendiculars on any tangent from two points on the minor axis, each distant $\sqrt{a^2 - b^2}$ from the centre, is $2a^2$.

13. Find the equation of the hyperbola referred to the asymptotes.

The axis CY revolves round the point C, while the axis CX is fixed and hyperbolas are described having CX and CY for asymptotes, and their foci at a given constant distance from C; prove that the extremities of their transverse axes lie on a circle.

14. Investigate the following loci:—

(1.) $(y - x)^2 - (y + x) = 1.$

(2.) $x^2 - \sqrt{5}xy + y^2 = 0.$

(3.) $\frac{1}{r} = 1 + \cos\theta + \sqrt{3}\sin\theta.$

PURE MATHEMATICS. (2.)

1. The base and the magnitude of the vertical angle of a triangle are given; show that the straight line bisecting the vertical angle passes always through a fixed point.

2. The first of six boys can copy 3 lines as soon as the second can copy 2; the second 5 as soon as the third 6; the third 7 as soon as the fourth 8; the fourth 9 as soon as the fifth 10; and the fifth 15 as soon as the sixth 14. How many lines will the sixth copy while the first is copying 135 lines?

3. The interior angles of an irregular polygon are in arithmetic progression; the least angle is 120° and the common difference 5°; find the number of sides.

4. If the chance that you can solve a certain problem is one-fifth, and the chance that two other particular persons cannot solve it between them is two-thirds, what is the chance that the problem will be solved among the three?

5. If the sides of a triangle be represented by

$$\frac{x}{y}+\frac{y}{z},\ \frac{y}{z}+\frac{z}{x},\ \frac{z}{x}+\frac{x}{y},$$

prove that its area $=\sqrt{\frac{x}{y}+\frac{y}{z}+\frac{z}{x}}$.

6. A straight line BC of given length slides with its ends on two fixed straight lines AX, AY, and BP, CP, the perpendiculars to AX and AY through B and C, intersect in P; prove that the locus of P on the plane XAY is a circle.

   Show also that the locus of P relative to the line BC is a circle of half the radius of the preceding circle.

7. Two circles intersect in A and B, and CBD is drawn perpendicular to AB to meet the circles in C and D; if EAF bisect either the interior or exterior angle between CA and DA, prove that the tangents to the circles at E and F intersect in a point on AB produced.

8. If A and B are two fixed points in the same plane with a given straight line, and on the same side of it, find the position of a point P on the straight line such that the sum of the distances AP, BP, may be the least possible.

   If AD, BE, CF be the perpendiculars let fall from the angular points of a triangle ABC on the opposite sides, prove that DEF is the triangle of least perimeter which can be inscribed in the triangle ABC.

9. Prove that

$(x+y+z)^5 - x^5 - y^5 - z^5$ is divisible by $(y+z)(z+x)(x+y)$, and determine the other factor.

10. Prove that

(1.) $\sec^2 \frac{\pi - a}{4} + \sec^2 \frac{\pi + a}{4} = \sec^2 \frac{\pi - a}{4} \sec^2 \frac{\pi + a}{4}$.

(2.) $\sec A + \sec (120° + A) + \sec (120° - A) + 3 \sec 3A = 0$.

11. If O be the centre of the inscribed circle of a triangle ABC, prove that

$$a \cdot AO^2 + b \cdot BO^2 + c \cdot CO^2 = abc.$$

12. Expand $(1+x)^{-\frac{1}{2}}$ in powers of x, writing down the coefficient of x^r; and prove that

$$1 - \frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{2^2} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{2^3} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1}{2^4} - \ldots = \frac{1}{\sqrt{2}}.$$

13. Find a number of six digits ending with 7, such that if it be multiplied by 3, the product is the same as the number obtained by transferring the first digit from the left to the right of the number.

14. Find the sums of n terms of each of the series

(α.) $\cos \theta + \cos 3\theta + \cos 5\theta + \ldots$

(β.) $\cos \theta \cos 2\theta + \cos 2\theta \cos 3\theta + \cos 3\theta \cos 4\theta + \ldots$

15. In how many ways can the crew of an eight-oar be made up out of twelve men, of whom ten can row and cannot steer, one can steer and cannot row, and one can both steer and row?

PURE MATHEMATICS. (3.)

1. If $f(x) = 0$ be an algebraic equation having two equal roots, prove that one root of $f'(x) = 0$ will be equal to either of them. Hence solve the equation $x^4 - 10x^2 + 8x + 8 = 0$, which has a pair of equal roots.

2. Show how to transform an equation into another whose roots shall be the reciprocals of those of the given equation.

   Solve the equation $6x^3 - 23x^2 - 6x + 8 = 0$, the roots of which are in harmonical progression.

3. How are the roots apparently obtained by Cardan's solution of the cubic equation $x^3 + qx + r = 0$ limited? Prove that the roots of $x^3 + px^2 + r = 0$ are all possible if either p or r be negative and $27r^2 + 4p^3r < 0$.

4. Determine the limit of $\frac{x}{a^x - 1}$, and of $\frac{\tan^{-1} x}{x}$, when x has zero for its limit; and obtain the differential coefficients of

$$f(x) \times \varphi(x) \text{ and } y^{x^2}.$$

5. If $\sec^{-1}\{\sqrt{(x-1)^2 - 1)e^x}\}$ be expanded by Maclaurin's theorem in ascending powers of x, the first terms are $\frac{\pi}{3} + \frac{1}{\lfloor 3} \frac{x^3}{\sqrt{3}}$.

6. Show how to obtain the limiting values of functions of one variable which take the indeterminate forms $\frac{0}{0}, \infty \times 0, 1^\infty$.

   Evaluate the functions

$$\frac{e^x - e^{-x}}{\tan x}, \left(\frac{\sin^{-1} x}{x}\right)^{\frac{1}{x^2}}, (\cot x)^{x^2},$$

   when $x = 0$.

7. Show that the length intercepted on the tangent of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ by the axes is a minimum when the tangent makes an angle $\tan^{-1}\sqrt{\frac{a}{b}}$ with the axis of y; and that its length is then $a + b$.

8. What is meant by a rectilinear asymptote to a curve?

   Show how to find the directions of the asymptotes to a plane curve whose equation is given.

9. Define the polar subtangent, and find an expression for its value in terms of r and θ.

Trace the following curves :—

(i.) $r\theta = a$;
(ii.) $r^2 = a^2 \cos 2\theta$;
(iii.) $ay^2 + y^3 + bx^2 = 0$.

10. Find an expression for the radius of curvature in terms of the radius vector and the perpendicular on the tangent. Find the radius of curvature at a point of the cardioid $r = a(1 - \cos\theta)$.

11. Find the envelope of the line $y = mx + c$, such that the perpendicular on it from a fixed point meets it on a fixed straight line.

12. Integrate the expressions

(i.) $\dfrac{x\,dx}{\sqrt{2ax - x^2}}$;

(ii.) $\dfrac{d\theta}{a + b\cos\theta}$;

(iii.) $\dfrac{dx}{1 + x + x^3}$;

(iv.) $\dfrac{dx}{x\sqrt{1 + x^3}}$.

13. A top consists of a right cone, whose height is equal to the radius of the base, surmounted by the segment of a sphere (described about the vertex of the cone as centre and with radius equal to the side of the cone) which is intercepted by the cone. Find the whole surface and volume of the top.

STATICS.

1. Show that, subject to certain assumptions, a statical force may be represented geometrically by a straight line.

   State the geometrical proposition upon which the compounding and resolving of forces depend. Assuming the truth of this proposition for two forces acting on a point so far as the *direction* of the resultant is concerned, prove it for the magnitude of the resultant.

   The wind makes an acute angle with the forward direction of a ship's course, and is received on a sail spread in a direction between that of the wind and that of the ship's course; explain generally by the resolution of force why the ship advances.

2. If any number of forces acting on a particle be represented in magnitude and direction by the sides of a polygon taken in order, the forces will be in equilibrium. Examine whether it is true that if the sides of the polygon be parallel to the forces, they will be in equilibrium.

   Four equal forces, P, Q, R, S, act in one plane on a point; the angles between P and Q, Q and R, R and S, are all equal, and the angle between P and S, 144°; find their resultant, and show how the condition of equilibrium of the four forces and the reversed resultant may be expressed by a polygon.

3. When are parallel forces said to be "like" and "unlike"? Define the centre of parallel forces; find it for three like parallel forces acting in one plane referred to rectangular co-ordinates; hence infer the similar expressions for the centre of any number of parallel forces.

   A, B, C, D are the angles of a parallelogram taken in order; like parallel forces 6, 10, 14, 10 act at A, B, C, D respectively; show that the centre and resultant of the parallel forces will remain unchanged if, instead of these forces, the parallel forces 8, 12, 16, 4 act at the points of bisection of the sides AB, BC, CD, DA respectively.

4. Define the centre of gravity of a solid, and prove that every body has one and only one centre of gravity.

   An inclined plane with an angle of inclination of 30° is 3 feet long; weights of 7, 5, 4, 8 ounces are placed on the plane in order, one foot apart, the weight 8 being at the top of the plane; find the distance of the centre of gravity of the weights from the base of the inclined plane.

   If the weights were attached to each other by fine threads, and the weight 8 was allowed to hang over the top of the plane, it being smooth, would the weights be at rest?

5. Find the centre of gravity of particles placed at the angular points of a triangle (i.) when the particles are equal, (ii.) when they are proportional to the opposite sides of the triangle.

6. Define a couple. Prove that, generally, any system of forces acting upon a rigid body in one plane may be reduced to a single force and a single couple. Hence define the conditions when the forces are in equilibrium.

7. Enunciate the principle of virtual velocities, and prove it when there is equilibrium in the system of three movable pulleys, where the strings are parallel and each attached to the weight; (usually called the third system).

   If (P) in this system be made to descend and draw up (W), show that the centre of gravity of (P) and (W) will neither ascend nor descend.

8. Describe either Roberval's or Quintenz's balance, and investigate its peculiar property.

9. A force (P) making an angle with a rough inclined plane is just on the point of moving a weight W up the plane; if (α) be the inclination of the plane, find the relation of P to W.

   If (P) act parallel to the plane, and the coefficient of friction be

$$\frac{\sin\left(45^\circ - \frac{\alpha}{2}\right)}{\sin\left(45^\circ + \frac{\alpha}{2}\right)},$$

   examine whether any mechanical advantage is obtained by the inclined plane.

10. What is a unit of work, and what is taken as the measure of a horse-power? When weights are raised through different heights, show that the whole work expended is equal to the work that would be expended in lifting a weight equal to the sum of the weights through the same height as that through which the centre of gravity of the weights had been raised.

    A tower is to be built of brickwork, the base is a rectangle 22 feet by 9, and the height of the tower is 66 feet, the walls being two feet thick; find the number of units of work expended in raising the bricks from the ground, and the number of hours in which an engine of three horse-power would raise them, a cubic foot of brickwork weighing one hundredweight.

DYNAMICS.

1. Describe clearly the difference between absolute and relative velocity, and state how the latter is determined.

2. Two particles move with velocities v and $2v$ respectively in opposite directions, in the circumference of a circle. In what positions is their relative velocity greatest and least respectively, and what values has it then?

3. Find the straight line of quickest descent from a given point to a given straight line.

4. A body is projected up an inclined plane, whose length is ten times the height of its upper end, with a velocity of 30 feet per second; in what time will its velocity be destroyed?

5. A body is thrown vertically upwards with a velocity of 56 feet per second from a height of 176 feet above the ground; find its velocity at the moment it reaches the ground.

6. A body is projected with a given velocity and at a given angle to the vertical: find the greatest height, and the time in which it reaches it.

7. A coach wheel, rolling with uniform velocity, throws off continually small particles of dust at a tangent to its circumference: find the position on the wheel from which the particles will rise to the greatest height. Explain the double result.

8. Explain fully the meaning of the formula $W = Mg$, defining the symbols, and stating how the equation is arrived at; also what units are usually implied when the formula is used.

9. Show that in the collision of two imperfectly elastic bodies the sum of the mass of each multiplied by the square of its velocity is diminished.

10. Having given the velocities of two bodies in any direction, find the velocity of their centre of gravity in the same direction.

11. A body weighing 2 lbs. is projected with velocity 20 at an angle of 60° to the horizon; another weighing 3 lbs. is at the same time projected from the same point at an angle of 30°, with a velocity 40. Find to two places of decimals the height to which their common centre of gravity mounts, and the distance at which it strikes the horizontal plane.

12. Find approximately the time of a small oscillation of a particle constrained to move in a vertical circle under the action of gravity.

13. A pendulum at sea level gains n seconds per day: to what height must it be elevated in order to keep true time?

14. Show how to make a pendulum oscillate in a given cycloid.

In what point of the cycloid will the force stretching the string, and the force accelerating the body, be equal?

15. Explain the dynamical meaning of the terms Work and Energy, and prove the relation which exists between them.

Find the energy per second of a waterfall 30 yards high and a quarter of a mile broad, where the mass of water is 20 feet deep, and has a velocity of $7\frac{1}{2}$ miles an hour when it arrives at the fall. The weight of the water is 1024 ozs. per cubic foot.

SET X.

EUCLID.

1. Define a plane rectilineal angle, a right angle, parallel straight lines, and a rhombus.

2. Bisect a given rectilineal angle.

3. If a side of a triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.

   ABC is a triangle, AD is drawn perpendicular to BC, and AE bisecting the angle A. Show that the angle DAE is half the difference of B and C.

4. If the square described on one of the sides of a triangle be equal to the squares described on the other two sides of it, the angle contained by these two sides is a right angle.

   ABC is an isosceles triangle, having the angles B and C equal: BD is drawn perpendicular to AC. Show that the difference of the squares on AD, CD is equal to the difference of the squares on AB, BC.

5. Divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts, shall be equal to the square on the other part.

6. If two circles touch each other externally in any point, the straight line which joins their centres shall pass through the point of contact.

7. If a straight line touch a circle, and from the point of contact a straight line be drawn meeting the circle, the angles which this line makes with the line touching the circle shall be equal to the angles which are in the alternate segments of the circle.

   Show that the angle in any segment of a circle is half the supplement of the angle contained by the tangents at the extremities of the base of the segment.

8. If from a point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square on the line which meets it, the line which meets shall touch the circle.

9. Describe a circle about a given triangle.

Show that the square on the side of an equilateral triangle is three times the square on the radius of the circumscribing circle.

10. Equal triangles, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional.

ABCD is a quadrilateral having the side AB parallel to CD. The diagonals meet in O. Prove that AO . OD = BO . OC (the corners A, C being opposite to one another).

11. Similar triangles are to one another in the duplicate ratio of their homologous sides.

ALGEBRA.

1. Prove that $a^m \times a^n = a^{m+n}$, m and n being positive integers; and assuming this formula to hold good for all values of m and n, determine the meanings that must be assigned to a^0 and $a^{\frac{1}{2}}$.

2. Given $x + y = \sqrt{m}$, and $x - y = \sqrt{n}$, express $x^3 + y^3$ in terms of m and n, and show that
$$16\,(x^4 - 7x^2y^2 + y^4) = (5m - n)\,(5n - m).$$

3. Divide $(m + 1)\,(bx + an)\,b^2x^2 - (n + 1)\,(mbx + a)\,a^2$ by $bx - a$.

4. Subtract $\dfrac{x + 3}{x^2 + x - 12}$ from $\dfrac{x + 4}{x^2 - x - 12}$, and divide the difference by $1 + \dfrac{2\,(x^2 - 12)}{x^2 + 7x + 12}$.

5. Find the greatest common measure of
$4x^4 - 13x^2y^2 + 9y^4$ and $10x^4 + 23x^3y - 28xy^3 - 15y^4$.

6. Show that the square root of the binomial $a + \sqrt{b}$ may be expressed as a binomial having one or both of its terms irrational, if $a^2 - b$ is a square number.

Find the value of $\dfrac{\sqrt{3 - \sqrt{5}}}{\sqrt{2} + \sqrt{7 - 3\sqrt{5}}}$ correct to three places of decimals.

7. Solve the equations

(1.) $(2x + 1.5)\,(3x - 2.25) = (2x - 1.125)\,(3x + 1.25)$.

(2.) $\dfrac{x}{2\,(x + 3)} - 2\dfrac{5}{24} = \dfrac{x^2}{x^2 - 9} - \dfrac{8x - 1}{4\,(x - 3)}$.

(3.) $\left.\begin{aligned} x^2\,(3ax - 2by) &= ab^3 \\ xy\,(3by - 2ax) &= a^2b^2 \end{aligned}\right\}$.

8. Prove that $\dfrac{b}{a}$ is the sum and $\dfrac{c}{a}$ the product of the roots of the equation $ax^2 - bx + c = 0$.

Show that the equation $x^3 - a^3 = 0$ has one real and two imaginary roots, and that the product of the two latter is equal to the square of their sum.

9. A wheel revolving with a given velocity drives a smaller wheel by means of cogs. If there had been 4 more cogs in each wheel, the velocity of the smaller wheel would have been one-sixth less, and if there had been 4 less in each, that velocity would have been one-third greater. Find the number of cogs in each wheel.

10. If $a : b :: c : d$, show that

$$a^2 + b^2 : ab :: c^2 + d^2 : cd.$$

The area of a certain rectangle is half that of the circumscribing circle. Prove that its sides are in the ratio $5 + \sqrt{3} : 5 - \sqrt{3}$, assuming that the area of a circle $= \frac{22}{7} \times (\text{radius})^2$.

11. If a, b, c, d be any four consecutive terms of an arithmetic series, show that $bc - ad$ is a positive quantity.

12. Investigate a formula for the sum of any number of quantities in Geometrical Progression, and explain when and in what sense an infinite series may be said to have a sum.

The first two terms of an infinite Geometrical series are together equal to 1, and every term is twice the sum of all the terms which follow. Find the series.

13. Assuming the formula for the number of Permutations of n things taken r together, deduce the number of Combinations of n things taken r together; and show both from the formulæ and from general reasoning that this number is the same as the number of Combinations of n things taken $n - r$ together.

A committee of 7 is to be chosen from 13 Candidates, of whom 6 are Liberals and 7 Conservatives. In how many ways can the selection be made so as to give a Liberal majority?

14. Expand $x^m \left(1 - \frac{1}{x}\right)^{-n}$ in a series of powers of x.

In the case in which m is a positive integer, write down the term which does not contain x, and prove that it will be a positive integer whenever n is a positive integer.

TRIGONOMETRY.

1. Examine the rule for transferring an angle expressed in French measure to the English scale.

   Express the angle of an equilateral and equiangular pentagon in the French scale, and thence determine its measure in the English scale.

2. Define as ratios the sine, and tangent of an angle, and show that any proper fraction, positive or negative, may be the sine of an angle, and any number, positive or negative, may be the tangent of an angle.

   Prove $\cos(90° - A) = \sin A = -\sin(-A)$.

3. If $(A + B)$ be less than a right angle, prove $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

   Deduce directly from this formula

   $$\text{Cos } 2A = (\cos A)^2 - (\sin A)^2.$$

   Express $\frac{1}{2}\{\cos(3A + 2B) + \cos(5A - 4B)\}$

   as the product of two cosines.

4. Find the sine and cosine of 30°, and prove

   $$\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}.$$

   Prove also $\sin 105° + \cos 105° = \cos 45°$.

5. Prove

   (1.) $$\text{Tan } A = \frac{2 \tan \frac{A}{2}}{1 - \left(\tan \frac{A}{2}\right)^2}.$$

   (2.) $1 + \tan 2A \,.\, \tan A = \sec 2A.$

   (3.) $$\text{Cos } 36° = \frac{\sqrt{5} + 1}{4}.$$

6. If π ($= 3{\cdot}1416$) be the ratio of the circumference of a circle to its diameter, of what angle is $\frac{\pi}{6}$ the circular measure? Express in degrees the unit of angle in circular measure, and show that this angle is subtended by an arc equal in length to the radius of the circle. Express in circular measure the least angle of an isosceles triangle in which the vertical angle is half of each of the angles at the base.

7. An observer on the top of a cliff 200 feet above the sea level observes the angles of depression of two ships at anchor in the same vertical plane with the observer to be 45° and 30° respectively; find the distance of the ships from each other.

8. When is the solution of an oblique-angled triangle said to be ambiguous?

   Given $\angle A = 30°$, $BC = 100$, $AB = 100\sqrt{3}$, solve both the triangles to which the data belong.

9. Given the three sides of a plane triangle, find an expression for the area of the triangle.

   Apply it to determine the area of the triangle whose three sides are ·9, 1·2, 1·5 respectively.

10. In any triangle whose sides a, b, c, subtend the angles A, B, C

    when $s = \frac{a+b+c}{2}$,

    Prove

    (1.) $\text{Tan}\,\frac{A-B}{2} = \frac{a-b}{a+b} \cdot \cot\frac{C}{2}$.

    (2.) $s(s-a) = (s-b)(s-c)$ if A is a right angle.

11. If Log represents the tabular logarithm,

    Given L sin 73° 20′ 10″ = 9·9813671

    L sin 73° 20′ 20″ = 9·9813784,

    Find L sin 73° 20′ 15·7″.

12. Find an expression for the radius (R) of the circle circumscribing a triangle. If r be the radius of the inscribed circle, show that

$$\frac{r}{R} = 4\left(\frac{s}{a}-1\right)\left(\frac{s}{b}-1\right)\left(\frac{s}{c}-1\right).$$

PURE MATHEMATICS. (1.)

1. If three straight lines meet all in one point and a straight line stands at right angles to each of them at that point, these three straight lines are in one and the same plane.

2. From the same point in a given plane there cannot be two straight lines at right angles to the plane upon the same side of it: and there can be but one perpendicular to a plane from a point above the plane.

   If a pyramid stand on a square base and have equilateral triangles for its faces, the perpendicular from the vertex upon the base is half the diagonal of the base.

3. Show that the square on the distance of any point within a cube from one of the angular points of the cube is equal to the sum of the squares on its distances from the three faces meeting at that angular point.

4. If a solid angle be contained by three plane angles, any two of them are greater than the third.

5. Define the terms conic section, focus, minor axis, asymptote.

   If a right circular cone be cut by a plane which is not parallel to a generating line, the semi-minor axis of the section is a mean proportional between the perpendiculars from the vertices of the section upon the axis of the cone.

6. If O be the centre and R the radius of the circle circumscribing a triangle, I the centre, and r the radius of an escribed circle, show that

$$OI^2 = R^2 + 2Rr.$$

   If D, E be the points in the base BC of the triangle ABC where the inscribed and escribed circles respectively touch it, prove that

$$BE = CD = s - c,$$
$$\text{and } BD = CE = s - b,$$
$$\text{where } s = \tfrac{1}{2}(a + b + c).$$

   Hence prove that $rr' = (s - b)(s - c)$ where r, r are the radii of the two circles.

7. Assuming De Moivre's theorem for the case of a positive integral exponent, prove that it holds also for negative and fractional exponents.

   Show that

$$\tan(\alpha + \beta + \gamma + \ldots) = \frac{S_1 - S_3 + S_5 - \ldots}{1 - S_2 + S_4 - \ldots},$$

   where S_r denotes the sum of the products of tan α, tan β, &c. r at a time.

8. Resolve $x^{2n} - 2x^n \cos\theta + 1$ into quadratic factors, and show that

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\ldots$$

9. Explain the meaning of the constants in the equations of a straight line in the forms

(1.) $ax + \beta y = 1$ (2.) $\frac{x-h}{l} = \frac{y-k}{m} = r.$

Find the distance of the point (hk) from the straight line represented by the equation

$$ax + \beta y + \gamma = 0.$$

10. Given the equations of two straight lines, viz.

$$ax + by + c = 0, \qquad a_1x + b_1y + c_1 = 0,$$

find the equations of the straight lines drawn through their point of intersection, and bisecting the angles between them.

11. Prove that the equation $x^2 + y^2 + ax + by + c = 0$ represents a circle, and find the co-ordinates of its centre and the length of its radius.

12. Find the equation of the common tangent to the two circles

$$x^2 + y^2 = c^2, \qquad \overline{x-a}\,|^2 + y^2 = b^2.$$

13. Find the equation of the tangent to a parabola, and prove either analytically or geometrically that the tangent at any point of a parabola is equally inclined to the axis and the focal distance of that point.

Express the equation to the normal in the form

$$y = m(x - 2a) - am^3.$$

14. Find the equations of the asymptotes of the hyperbola whose equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

PURE MATHEMATICS. (2.)

1. Find the value of $x^3 + 3a^2x + 4a^3$, when

$$\frac{2x}{a} = 1 + \sqrt{-15}.$$

Simplify $\sqrt{17 + 20\sqrt{-2}} + \sqrt{17 - 20\sqrt{-2}}$.

2. Roman silver coin contains $\frac{29}{30}$ parts of pure silver, and English silver $\frac{37}{40}$ parts: what will be the value of 222 denarii of 60 grains each, at the rate of 5*s.* 2*d.* per oz. standard?

3. If in a given time A can do $\left(\frac{1}{p}\right)^{\text{th}}$ part of the work that B and C can do together, B $\left(\frac{1}{q}\right)^{\text{th}}$ of what C and A can do together, and C $\left(\frac{1}{r}\right)^{\text{th}}$ of what A and B can do together; compare the times that each would require to perform separately a fixed amount of work.

4. Develop $(1 - ax)^{-1}(1 - bx)^{-1}$ in a series of ascending powers of x; and find the sum of n terms of the series.

5. Three cards are taken at random from an ordinary pack; find the chance that they will consist of a knave, a queen, and a king.

6. There are two numbers whose product is 192, and the quotient of the arithmetical by the harmonical mean of their greatest common measure and least common multiple is $3\frac{25}{48}$: find the numbers.

7. Sum to n terms

$$2 + 5 + 8 + 11 + \ldots$$
$$2 \times 5 + 5 \times 8 + 8 \times 11 + \ldots$$
$$\cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x + \ldots \text{ to infinity.}$$

8. Find the general formula for all angles possessing the same sine as a given angle.

Solve the equation $2 \sin \alpha \sin 3\alpha - \sin^2 2\alpha = 0$.

9. Establish the following :—

$$\frac{1 + \cot 60°}{1 - \cot 60°} = \left\{ \frac{1 + \cos 30°}{1 - \cos 30°} \right\}^{\frac{1}{2}}$$

$$\sin^{-1}\left(\frac{x - a + \beta}{2\beta}\right)^{\frac{1}{2}} = \frac{1}{2}\cos^{-1}\frac{a - x}{\beta}$$

and that in any plane triangle

$$\log a = \log (b - c) + \mathrm{L}\cos\frac{\mathrm{A}}{2} - \mathrm{L}\cos\phi,$$

where $\phi = \tan^{-1}\left\{\frac{b + c}{b - c}\tan\frac{\mathrm{A}}{2}\right\}$.

10. Given the n^{th} part of a straight line, find the m^{th} part of the same straight line by a geometrical construction.

11. If an equilateral triangle, a square, and a regular hexagon be described about the same circle, their sides are in a geometrical progression.

12. If semicircles ADB, BEC be described on the sides AB, BC of a right-angled triangle, and on the hypotenuse another semicircle AFBGC be described, passing through the vertex B, the lunes AFBD and BGCE are together equal in area to the triangle ABC.

13. Given a circle, and a point without it, describe a square about the circle in such position that the straight line joining two adjacent points of contact shall when produced pass through the given point.

14. Obtain the equation to a straight line in the form $x \cos \alpha + y \sin \alpha - p = 0$, where p is the perpendicular on the line from the origin, and α the inclination of that perpendicular to the axis of x.

Draw the straight lines

$$x + y\sqrt{3} = 6 + 2\sqrt{3},\ x - y\sqrt{3} = 6 - 2\sqrt{3};$$

and find the equations to the straight lines bisecting the angles between these two.

15. Through a point O without a circle a straight line is drawn, meeting the circle in A and B, and also meeting the line joining the points of contact of the tangents drawn from O in C. Show that

$$2\,\mathrm{OA}.\mathrm{OB} = \mathrm{OC}\,(\mathrm{OA} + \mathrm{OB}).$$

PURE MATHEMATICS. (3.)

1. If two values give different signs to $f(x)$ when substituted for x, prove that a root of the equation $f(x) = 0$ must lie between them.

   Prove that the equation

   $$(x - 1)e^x - (x + 1)e^{-x} = 0$$

   has one root at least between 1 and 2.

2. Find the coefficients in terms of the roots of an equation.

   The roots of the equation

   $$8x^3 - 14x^2 + 7x - 1 = 0$$

   are in geometrical progression; find them.

3. An odd number of roots of the equation $f'(x) = 0$ must lie between each adjacent pair of real roots of $f(x) = 0$.

   Hence prove that the equation in Question 1 has only two real roots, one positive, the other negative.

4. Obtain the differential coefficients of the following functions:—

   $$(1)\ x^n;\ (2)\ \tan x;\ (3)\ \tan^{-1}\frac{\sqrt{x} - \sqrt{1-x}}{\sqrt{x} + \sqrt{1-x}}.$$

5. If $\tan y = \cos x$, expand y in terms of x to 4 terms.

6. If $y = \sin(x + a) + \dfrac{\cos(x + a)}{x}$,

   prove that $\dfrac{d^2y}{dx^2} + \left(1 - \dfrac{2}{x^2}\right)y = 0.$

7. Find the conditions that $x = a$ should make $f(x)$ a maximum or minimum.

   A length l of wire is cut into two portions, which are bent into the figures of a circle and a square respectively: how must it be divided in order that the sum of their areas may be the least possible?

8. Find the asymptotes of the curve

   $$y^2(x - a) = x^2(x - 2a).$$

9. Trace the curve given in Question 8, and also the curve

   $$r^2 = \frac{c^2}{(a - \theta)^2}.$$

10. In plane curves prove that $\frac{dr}{ds} = \sqrt{1 - \frac{p^2}{r^2}}$.

If $s = a \sin^{-1}\frac{p}{r}$, and if r and p vanish together, prove that $r^2 = ap$.

11. Prove that the radius of curvature is $r\frac{dr}{dp}$.

Hence prove that the curve whose equation is given in Question 10, must be a circle.

12. Integrate

(1.) $\sqrt{a^2 - x^2}dx$.

(2.) $\frac{dx}{(x-4)^2(x^2+5x+6)}$.

(3.) $(\log \sin x + x \cot x)\, dx$.

13. Find the area of a segment of a parabola cut off by the latus rectum.

STATICS.

1. Mention any fundamental principles by means of which the science of Statics can be treated geometrically. Can a body be at rest if a single force only act upon it? What statical forces are in action, (1) when a weight is supported at rest hanging by a string, (2) when a weight is at rest on a rough inclined plane?

   When two forces act on a particle, prove the parallelogram of forces so far as the direction of the resultant is concerned, and by the extension of the properties of the parallelogram of forces show how the resultant of any number of forces acting on a particle may be found.

   Two equal forces act on a particle; find the angle at which they act, when the square of their resultant is equal to three times their product.

2. Define a lever. What is the moment of force about a given point?

   A straight lever without weight is suspended by a string and sustains in equilibrium two weights attached to the opposite extremities of the lever; investigate all the conditions of equilibrium. Is it necessary for equilibrium that the lever be horizontal?

   The pressure on the fulcrum of two weights in equilibrium when suspended at the extremities of a straight lever 12 inches long is 20 pounds, and the ratio of the distances of the fulcrum from each extremity of the lever is 3 : 2; find the weights.

3. Three unequal heavy particles are at given vertical heights above a horizontal plane; find the height of their centre of gravity above the plane.

   A triangular pyramid is supported with its vertex on a horizontal plane and its altitude, perpendicular to its base, vertical; equal heavy particles are placed in each of the angles of the pyramid; find the height of the centre of gravity of all the particles above the horizontal plane.

4. Prove that the algebraical sum of the moments of two forces round a fixed point *within* the angle formed by the directions of the forces and in the same plane with them, is equal to the moment of the resultant.

   AB is the diameter of a circle, BP and BQ are chords at right angles to each other; if BP and BQ represent forces, prove that their moments about A are equal.

5. When are couples said to be like and when unlike? When will two unlike couples balance each other? (1) If a system of forces is represented in magnitude and position by the sides of a plane polygon taken in order, show that the system must be equivalent to a couple. (2) If the sides of a parallelogram taken in order represent a system of forces acting upon a body, express the moment of the couple to which the system of forces is equivalent.

6. How may the thread of a screw be traced on a cylinder? Find the equilibrium relation between the power and the weight in a screw as ordinarily considered, friction being neglected.

7. State the principal elementary laws of statical friction.

   Two equally rough inclined planes CA and CB are placed back to back having a common vertex C, AB being horizontal, (α) and (β) the inclinations of the planes; weights (P) and (W) connected by a fine thread passing over C are placed on the planes CA and CB respectively; if P be on the point of sliding down CA, prove $P \sin(\alpha - \phi) = W \sin(\beta + \phi)$, where ($\phi$) is the limiting angle of resistance for both planes.

8. State the principle of "virtual velocities," and show that it holds good when a weight (W) is supported on a smooth inclined plane by a force (P) making an angle (β) with the plane, the inclination of which is (α). If the weight (W) on a plane be supported by another weight (P) attached to it by a thin string and hanging over the top of the inclined plane, and if (P), when in the same horizontal line with (W), be made to descend through a given height, show that the centre of gravity of (P) and (W) will move in a horizontal line, as long as (W) remains on the plane.

9. When the centre of gravity of a heavy body is given, and also the centre of gravity of any part of it, show how to find the centre of gravity of the remainder.

   ABCD is a rectangle, find a point P in AD so that when the triangle PDC is taken away the remaining trapezoid ABCP when suspended from P may hang with its sides AP and BC horizontal.

10. In the theory of work, what is meant by the modulus of a machine? The horse-power of a machine is usually taken at 33,000: explain the units referred to. How is the work done by a water-wheel moved by a fall of water usually estimated? The mean section of a stream is 5 feet by 2, its mean velocity is 35 feet per minute, there is a fall of 13 feet on a water-wheel whose modulus is ·65; determine the horse-power of the wheel.

DYNAMICS.

1. How is velocity measured, (1) linear, (2) angular?

   If the minute hand of a clock be six inches long, find the linear velocity of its extremity, in the ordinary units, π being $\frac{22}{7}$.

2. How are velocities compounded?

   The radius of the wheel of a railway carriage is 2 feet, and it makes $3\frac{1}{2}$ revolutions per second when the train is moving at 30 miles an hour: if π be $\frac{22}{7}$ find the actual velocity at any instant (1) of the point in contact with the ground; (2) of either of the two points one yard above the ground.

3. What is meant by saying that the acceleration of gravity is 32?

4. Find the time required to describe a given space from rest with uniform acceleration.

5. A balloon is 432 yards high, and is moving horizontally with the wind at the rate of 4 miles an hour. Find the path of a body dropped from the balloon (1) relative to the balloon, (2) in fixed space. Whereabouts does the body reach the ground?

6. If AC be the height and AB the length of a smooth inclined plane, prove that the time of sliding down it varies as $\frac{AB}{\sqrt{AC}}$.

7. If (using the construction of the last question) two equal bodies start simultaneously from A along AB and AC respectively, and if CD be perpendicular from C on AB, prove that their centre of gravity describes the line drawn from A to the middle part of CD with uniform acceleration.

8. A railway carriage slides freely down a smooth incline. Prove that if a weight be suspended by a string from the roof it will hang with the string perpendicular to the floor. If the angle of incline be α and the weight be W, find the tension of the string.

9. Find the velocities of two elastic spheres after direct impact; and deduce the velocity of an elastic sphere after impact on a plane.

10. Two balls are dropped simultaneously from points not in the same vertical line, and whose heights above a perfectly elastic horizontal plane are m^2 and n^2 feet respectively; prove that if M be the Least Common Multiple of m and n the balls will be in their original positions after successive intervals of $\frac{M}{2}$ seconds. ($g = 32$.)

11. Find the acceleration of a particle describing a circle with uniform velocity.

    What is the acceleration when the velocity is variable?

12. If, in Question 2, a particle weighing 1 oz. be attached to a point of the circumference of the wheel, find the magnitude of the attaching force when the particle is (1) at its highest point, (2) in the horizontal line through the centre.

13. How is work measured? What is meant by the Conservation of Energy?

    Taking the ordinary unit, how much energy is there in a body weighing 1 oz. moving at the rate of 30 miles an hour?

14. Find the time of oscillation in a smooth inverted cycloid, and hence the time of small oscillation of a pendulum.

    What would be the time of small oscillation of the suspended weight mentioned in Question 8, the string being one yard long, and the inclination of the plane 30°?

15. A seconds pendulum is lengthened by heat by the 100th part of itself; how many seconds does it lose in a day?

ANSWERS.

SET I.

ALGEBRA.

1. $\frac{3}{2}$. **2.** $x^3 - 4y^3 - 9z^3 - 12(yz)^{\frac{3}{2}}$. **3.** $4(x+y+z)$.

4. $(a-b)(b-c)(c-a)$. **5.** $\dfrac{x(a-b)-y(a+b)}{x(a-b)+y(a+b)}$.

7. $\dfrac{16+5\sqrt{7}-\sqrt{5}\sqrt[4]{7}.(5+\sqrt{7})}{9}$.

8. (i). $\dfrac{ab-cd}{a+b-c-d}$. (ii). 0, 9; 0, 25. (iii). 3, $-\frac{2}{15}$. (iv). 3, 4, 5.

9. $P = p^4 - 4p^2q + 2q^2$. $Q = q^4$. **10.** 90 days. 180 days.

11. 14 and 32 feet. **12.** $\frac{2}{5}$, $\frac{1}{2}$, $\frac{2}{3}$, 1. 12 hours. 1.30 A.M.

13. 12250.

TRIGONOMETRY.

6. 90°. **8.** 1500 square feet. **9.** 90°.

10. 3·2 miles an hour nearly. **11.** 84 feet.

12. 5, 16, $5\sqrt{15}$, 12 feet. 128° 39′ 13″, 51° 20′ 47″, 112° 23′ 37″, 67° 36′ 23″.

PURE MATHEMATICS (1).

12. $\tan^{-1}\dfrac{A}{B}$. **13.** $-\dfrac{C}{\sqrt{A^2+B^2}}$, length of perpendicular from origin on straight line.

PURE MATHEMATICS (2).

8. £2311·4. **9.** $\sqrt{10} = 3 + \dfrac{1}{6 + \frac{1}{6}\ldots}$, $\frac{3}{1}$, $\frac{19}{6}$, $\frac{117}{37}$, $\frac{721}{228}$.

10. 7, 5. **11.** 2, 3, 6, 11; 12, 7, 4, 3.

12. (α). $\dfrac{n(n+1)}{6}\dfrac{(n+2)}{}$. ($\beta$). 4. **14.** $\dfrac{1}{3^3} - \dfrac{1}{4^3}$. $\dfrac{1}{3^{r+1}} - \dfrac{1}{4^{r+1}}$.

15. 12. £14. 18*s*. **16.** $\frac{5}{36}$.

Pure Mathematics (3).

1. $-2, -1$. **5.** (1). $\frac{x^2-1}{x^2-4}$. (2). $\frac{\sin x}{1+\cos^2 x}$: $\frac{y^{\frac{1}{2}}(\log x - 2)}{x^{\frac{1}{2}}(\log y - 2)}$.

8. $1 + \frac{x}{e} + \frac{x^2}{e^2 \lfloor 2} - \frac{x^3}{e^3 \lfloor 3} - \frac{9x^4}{e^4 \lfloor 4} - \frac{6x^5}{e^5 \lfloor 5} + \ldots$. **10.** 0, 1.

13. $y = x + 4a = -x - \frac{4a}{3} = 2x - \frac{5a}{3}$. Approximates to the two straight lines $y = \pm 3x$.

14. $\log \frac{x-3}{x-2}$. **15.** $\frac{\pi a^3}{2}$.

Statics.

1. P making angle of 45° with larger force.

7. $3\frac{1}{3}$ inches from end at which 1 oz. hangs.

8. $\frac{W3\sqrt{5}}{5}$. $\frac{W\sqrt{10}}{5}$. **9.** $12\frac{12}{13}$. **10.** 1500. **11.** $\frac{28}{3\pi}$ lbs.

Dynamics.

1. 320. **3.** Parts are in ratio 1 : 3 : 5. 3″. **6.** 15 lbs.

7. 40 lbs. **9.** 77. **10.** 554 nearly.

SET II.

Algebra.

2. $16x^2(1-4x^2)$. $x^4 - 2bx^3 - x^2(a^2-b^2) + 2a^2bx - a^2b^2$. 0.

3. $x^2 + x(a-b) - ab$. 1.

4. $x^{\frac{2}{2n+1}} y^{\frac{2n-2}{2n+1}} - x^{\frac{1}{2n+1}} y^{\frac{2n-1}{2n+1}} + y^{\frac{2n}{2n+1}}$. $2n+1$.

5. $\frac{1+3x^{\frac{1}{3}}}{1-x^{\frac{1}{3}}}$. **6.** (i). $8\frac{2}{7}$. (ii). 3, $-\frac{1}{5}$. (iii). $\frac{ab}{2c}, 2d$. (iv). 7, $-\frac{29}{8}$; 2, $-\frac{13}{2}$; 4, $\frac{49}{8}$. **7.** $x^2 + 10x + 61 = 0$. $\frac{p_1q - pq_1}{q - q_1}$.

8. 58 sovereigns, 42 half-crowns. **9.** 10 gallons. **10.** 360.

11. $3 + 9 + 27 + \ldots$ or $48 - 36 + 27 - \ldots$. $\frac{1 - br + br^2}{(1-br)(1-r)}$.

14. 5th and 6th. $\frac{9375}{16}$.

Trigonometry.

1. 1·309 feet. **11.** 332.

Pure Mathematics (1).

3. $\frac{5}{6}$ nearly. **7.** (1). $\frac{\sin\frac{n+1}{2}\theta \sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$. (2). $\frac{\cos\frac{n+1}{2}\theta \sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$.

9. $-\sqrt{2a^2-b^2}$, $-\frac{b^2}{\sqrt{2a^2-b^2}}$. **12.** $\sqrt{2}, \sqrt{3}, \pi\sqrt{6}$. **13.** $\sqrt{2}$.

PURE MATHEMATICS (2).

7. $1 + 2x + 3x^2 + 4x^3 + \ldots$. **8.** $\frac{n}{5n+1}$.

9. 2, 8, 17, 5, 11, 3, 1 : 11, 3, 1, 5, 2, 8, 17.

10. First four convergent $\frac{3}{1}, \frac{4}{1}, \frac{11}{3}, \frac{15}{4}$. 1_6

PURE MATHEMATICS (3).

2. $x^3 - \frac{9x^2}{2} + \frac{13x}{2} - \frac{15}{4} = 0$. **5.** $-\frac{2a^2x}{a^4+x^4}$, $\frac{2\sin x}{\cos x}$.

7. $\frac{m}{m+n}$. **8.** $-\frac{1}{2}, 2, \frac{AB}{BD}$. **11.** $ae^{n\theta}\sqrt{1+n^2}$.

12. $x\log x - x$. $\frac{-x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$: $\frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32}$.

13. $\log\frac{(x-1)^{\frac{3}{4}}(x+2)^6}{(2x+3)^{\frac{33}{4}}}$. **14.** $\sqrt{ax+x^2} + a\log\frac{\sqrt{x}+\sqrt{a+x}}{\sqrt{a}}$.

STATICS.

2. 120 lbs. **5.** $9\frac{27}{32}$ hours. **6.** $12\sqrt{2}$ stone.

9. If $2a$ = length of beam, c = distance of prop from wall, θ = inclination of beam to horizon,

$$\cos^3\theta = \frac{c}{a}\cdot\frac{W+W'}{W+2W'}.$$

11. $R = 4r$. **12.** $3\sqrt{3}\,.\,\pi a^2 b$, $12\pi ab$.

DYNAMICS.

1. $\frac{a^2}{b}, \frac{at}{b}$. 289800 feet. **8.** Masses form a G.P. of common ratio $\frac{1}{\epsilon}$: $\epsilon^{s-1}v$, where v is original velocity of first ball.

9. 6 . weight of particle. **10.** $\sqrt{l_1 l_2}$. $\frac{(3600)^2 - n^2}{(3600)^2}$.

11. $\frac{24}{5\,4}$ mile. 5″.

SET III.

ALGEBRA.

2. $(x^3-1)a^3 - a^2(x^3+x^2-2) + a(4x^2+3x+2) - 3(x+1)$.
$x^3 - 3x^2y - 3xy^2 + y^3$.

3. $3a^2 - ab + 5b^2, \frac{b^2}{4a}$. **4.** $\frac{2px}{3} + r$.

5. (1). $\frac{1}{x+y}$. (2). $(x^5\,.\,y^{13})^{\frac{1}{17}}$. **7.** (i). 6. (ii). $a, b-a$.

10. 30 miles.

Trigonometry.

2. $2n\pi \pm \frac{\pi}{4}, \frac{1}{4}\left(2n\pi \pm \frac{\pi}{4}\right)$. **3.** $\pm \frac{1}{\sqrt{2}}, \pm \frac{\sqrt{3}}{2}$. $45°, 60°, 75°$.

4. $\frac{7}{5\sqrt{2}}$. **7.** 4028·5, 2831·7. **9.** $\frac{c(1-\sin a)}{\cos a} + h$, where h is given height.

Pure Mathematics (1).

4. $\frac{a^3\sqrt{2}}{3}, \frac{\pi a^2\sqrt{2}}{3}$. **6.** $\cos\frac{2\pi}{5} + \sqrt{-1}\,.\sin\frac{2\pi}{5}$, &c.

8. (α). $e^{-\cos\theta}\sin(\sin\theta)$. ($\beta$). $\frac{\pi^2}{6}$. (γ). $\frac{\pi^2}{12}$.

12. $mm' + 1 = 0$. $(b-b')(2y-b-b') + (a-a')(2x-a-a') = 0$.

13. $c = \frac{a}{m}$. Rectangular hyperbola, semi-axis of which is $2a\sqrt{2}$.

14. An ellipse, semi-axes are $\sqrt{\frac{3-\sqrt{5}}{2}}, \sqrt{\frac{3+\sqrt{5}}{2}}$.

15. $Yx + Xy = 2a^2$.

Pure Mathematics (2).

7. $(1-r)^{-2}$. $\frac{n\,.\,(n+1)}{2}$. **9.** $m+1$.

Pure Mathematics (3).

2. $2p$. $3r$.

5. $\frac{a-2x}{(x^2-xa+a^2)^2}$. $\frac{1}{1+x^2}$. $(1+x^2)^{\frac{x}{2}}\left\{\frac{x^2}{1+x^2} + \frac{1}{2}\log(1+x^2)\right\}$.

7. $\pm 1, \mp 4$. **8.** $n^2, n^3 - n, n^4 - 4n^3$.

9. At a distance a from the origin and perpendicular to the initial line.

11. $\log\frac{x+1}{x+2}$. $\frac{1}{3}\sin^{-1}\frac{x^3}{a^3}$. $\log(\sin x)$. $\frac{1}{8}\sin 3x + \frac{1}{2}\sin x$.

12. $\frac{3\pi a^4}{8}$. **13.** (α). $\frac{2a^2}{3}$. (β). $\frac{a^2}{2}$.

Statics.

10. 6272 units.

Dynamics.

1. $4\frac{1}{2}''$. **6.** An ellipse. **8.** $\frac{\pi}{4}$.

9. When it has passed over a vertical distance $\frac{1}{3}$ radius.

10. $\frac{2\pi v}{v+v'}$.

SET IV.

Algebra.

1. $x^2y^2(a^4+b^4)-a^2b^2(x^4+y^4)$. $2(a^2+b^2)(x^2+y^2)$. $7\dfrac{(x^2+a^2)}{y^2+b^2}$.

2. $x^3-3x^2+11x-16$. $x^2+y^2+z^2-xy-yz-xz$.

4. $x-15$. **5.** $\frac{1}{2}$. **6.** $x^2+2xa-a^2$. $x^3-11x+17$.

7. (α). 12. (β). 5, $2\frac{11}{12}$. (γ). $a+b$, $\dfrac{ab(a+b)}{a^2+b^2}$. (δ). ± 4; ± 3.

8. £600, £900. **9.** £130, £260. **12.** $4n^2b+2n(a+b)+a$.

13. $n\,.\,2^{n+2}-2^{n+1}+2$. **15.** $\dfrac{3^5}{2^{\frac{19}{3}}}$.

Trigonometry.

1. $\dfrac{225^\circ}{11}$. **2.** 1, $\frac{1}{2}$. **7.** $n\pi+\dfrac{\pi}{4}$.

8. 105° 38′ 57″. 15° 38′ 57″. **12.** ·555 square foot.

Pure Mathematics (1).

3. $\dfrac{3V}{\Sigma(A)}$. $\dfrac{h}{\sqrt{3}-1}$.

6. (1). $\dfrac{\sin\left\{\alpha+\dfrac{(n-1)(\pi+\beta)}{2}\right\}\sin\dfrac{n(\pi+\beta)}{2}}{\sin\dfrac{\pi+\beta}{2}}$.

(2). $\dfrac{\sin^{n+1}\beta\sin n\alpha-\sin^n\beta\sin(n+1)\alpha+\sin\alpha}{\sin^2\beta-2\sin\beta\cos\alpha+1}$.

11. $\dfrac{m(a-1)-l(y-b)}{\sqrt{l^2+m^2}}=\pm\dfrac{m'(x-a)-l'(y-b)}{\sqrt{l'^2+m'^2}}$.

13. $\sqrt{\dfrac{a^2(a^2-r^2)}{a^2+b^2-r^2}}$.

Pure Mathematics (2).

10. $\dfrac{3^{n+1}-3-2n}{2^2\,.\,3^{n-1}}$. $\dfrac{3^{n+1}-2n-3}{2^2\,.\,3^{n-1}}-\dfrac{n\,.\,(n+1)}{2\,.\,3^n}$.

11. 8. **12.** $\dfrac{2}{3ab+2}$, $\dfrac{3a}{3ab+2}$.

Pure Mathematics (3).

3. 1, $\dfrac{-1\pm\sqrt{-3}}{4}$. $\dfrac{3\pm\sqrt{5}}{2}$. **4.** Divergent. **6.** $\dfrac{-24}{x^2}$.

7. $x^3 + x^3 + \frac{2x^4}{3}$. **9.** Minimum when $x = -3$.

10. $y + 2x = 0$, $y = \pm x$. **11.** $\frac{b^2}{a}$. **12.** $27ay^2 = 4(x - 2a)^3$.
Evolute is semi-cubical parabola.

13. $\log \frac{(x+2)^2}{x+1}$. $e^x(x-1)$. **14.** $r = Ke^{\frac{\theta}{\sqrt{e^2-1}}}$.

STATICS.

2. If F be horizontal force, $\theta = \tan^{-1}\left(\frac{F}{W}\right)$. **4.** $\frac{s}{2^n}$.

3. Angle between strings is 120°. $W\sqrt{3}$.

5. Angle between string and wall is 30°. $T = \frac{2W}{\sqrt{3}}$. $R = \frac{W}{\sqrt{3}}$.

7. $\frac{W}{\sqrt{3}}$. **9.** $\frac{3}{8}h$ from base. **11.** 7392000 foot-pounds. $7\frac{1}{15}$.

DYNAMICS.

1. $\sqrt{(a-ut)^2 + u^2t^2}$, a being distance of station from junction, u = velocity of each train. **8.** $\tan^{-1}\left(\frac{v^2}{rg}\right)$.

9. $g\cos\frac{\theta}{2}$, θ being angle through which generating circle has passed.

10. $\left(\frac{8641}{8639}\right)^2$.

SET V.

ALGEBRA.

2. ab. **3.** $1 - 4x - \frac{46x^2}{15} + \frac{10x^3}{3}$. $(x-12)(x-7)$.
$(x^2 + ax + a^2)(x^2 - xa + a^2)$. **4.** $(x-a)^3$.

6. $14 + 2\sqrt{21}$. **7.** (α). 2. (β). 4, $\frac{2}{19}$. (γ). ± 3, ∓ 3.

8. $4x^2 - 16x + 9 = 0$. **9.** (α). 357. (β). $6\frac{1}{4}$.

10. 90. **11.** $\frac{n.\overline{n-1}....\overline{n-r+1}}{\lfloor r}$. $r+1$. $\frac{1.3.5....(2r-1)}{2^r \lfloor r}$.

12. 4 gallons. 3 gallons. **13.** 4′.

TRIGONOMETRY.

1. $49\frac{1}{11}$ miles. **4.** None.

10. 2·529823. **11.** 2·974 square feet.

PURE MATHEMATICS (1).

3. One, if the points be not at the ends of a diameter.

7. $\frac{\sin 2n\beta}{\sin\beta}\cos\{\alpha + (2n-1)\beta\}$. $\frac{\sin n\beta\cos\{\alpha + \overline{n-1}\,\beta\}}{\sin\beta}$.

9. If A is taken as pole, required equation is

$$r^2 - 4cr\cos\theta + c^2(1+\cos^2\theta) = 0.$$

10. Straight line. **11.** Circle.

PURE MATHEMATICS (2).

3. $32\frac{8}{11}'$ after 3. **4.** $x^2+y^2+z^2+xy+yz+xz$. **5.** 9.
9. 210. **10.** 2160, 1300. **12.** 1·991226. ·544068.
14. (1). $b = c$. (2). $(b+c)^2 = 2a^2$.
17. $2a_n = a+b+\frac{a-b}{3^n}$. $2b_n = a+b-\frac{a-b}{3^n}$.

PURE MATHEMATICS (3).

1. Four imaginary roots. **2.** $-\frac{p_{n-1}}{p_n}$. $\frac{p^2_{n-1}-2p_n\cdot p_{n-2}}{p_n^2}$.

3. 6. **5.** $\cot x$, $\frac{\cos(\log x)}{x}$, $\frac{3(x^2-1)(1-x+x^2)^{\frac{1}{2}}}{(1+x+x^2)^{\frac{1}{2}}}$. **6.** r, r^2.

7. $\pm\frac{a}{\sqrt{a^2+b^2}}$, $\frac{\pm b}{\sqrt{a^2+b^2}}$. **8.** One negative root lying between 0 and -1.

9. (1). Maximum if $x = 1$. Minimum if $x = -1$. (2). Minimum if $x = \frac{7\pi}{6}$.

10. $1+x-\frac{2x^3}{\underline{3}}-\ldots.+\frac{2^n}{\underline{2}}\cos\frac{n\pi}{4}.x^n+\ldots.$ **11.** $x\pm a = 0$. $y = x$.

13. $\sin x$. $\frac{1}{2}\sin x^2$. $\frac{1}{a}\tan^{-1}\frac{x}{a}$. $\log(x^2+a^2)$.

$x\sin x+\cos x$, $\frac{1}{4a^3}\log\frac{x-a}{x+a}-\frac{1}{2a^3}\tan^{-1}\frac{x}{a}$. $\frac{1}{\sin a}\tan^{-1}\left(\frac{\sin x}{\sin a}\right)$.

14. $2a^2$.

STATICS.

10. 1120 units of work. $2\frac{2}{5}''$.

DYNAMICS.

1. $\frac{v}{64}$. **2.** 960 feet. **5.** $2en\sqrt{3}(1+e)$. **8.** 4900 feet.

9. $\frac{u^2v\cos a \pm uv\sqrt{v^2-u^2\sin^2 a}}{u^2-v^2}$, where u, v are velocities of fall and object, a the angle which direction of motion of object makes with line joining initial position of object to man.

11. $4\sqrt{5}$. A straight line.

SET VI.

ALGEBRA.

1. $\dfrac{ax(x^2 - xa + a^2)}{(a-x)^2(x^2 + xa + a^2)}$. **4.** $(x-1)^2$. $a^4 - b^4$.

5. (1). 7. (2). $\pm\sqrt{2}, \pm\sqrt{3}, \pm\sqrt{6}$. (3). $\dfrac{b}{a-b}, -\dfrac{a}{a+b}$.

6. $-\frac{1}{2}$. **7.** $3+\sqrt{7}$. **8.** 101211. **10.** $290\frac{2}{3}$. **11.** 297000.

12. $\dfrac{\lfloor\underline{16}}{(\lfloor\underline{8})^2}(ax)^8$. **13.** 40 and 45 miles an hour.

TRIGONOMETRY.

1. 1° 26′ nearly. **5.** $n\pi \pm \dfrac{\pi}{3}$. **10.** 33° 29′ 30″. 101° 30′ 30″.

PURE MATHEMATICS (1).

3. At angular points of an equilateral triangle on a great circle through them.

5. $\cos^2\dfrac{\pi}{n}$. 6. **9.** -5. **10.** $x^2 + y^2 = 9$.

11. $x^2m^2 + 2x(mc - 2a) + c^2 = 0$.

PURE MATHEMATICS (2).

2. $-\frac{31}{120}, \frac{319}{120}, \frac{481}{120}$. **3.** 3 miles an hour.

6. $\dfrac{(2n+1)\pi}{24}$ or $\dfrac{n\pi}{4}$. **11.** 142857. **12.** $\dfrac{\pi}{10}$ and $\dfrac{3\pi}{10}$.

13. 4, -4, $\frac{3}{2}$. 18. **14.** $\dfrac{r(2+\sqrt{3})}{\sqrt{3}}$. **15.** 3.

PURE MATHEMATICS (3).

1. $x^4 + 2x^2 + 25 = 0$. **2.** $x^5 + 11x^4 + 42x^3 + 57x^2 - 13x - 60 = 0$.

4. $\sin x \cos 2x \tan 3x\left(\cot x - 2\tan 2x + \dfrac{8\cos 2x}{\sin 2x \cos^2 3x}\right)$.

5. Maximum if $x = a$. Minimum if $x = \dfrac{5a}{3}$. **6.** $\dfrac{\pi}{3}$ and $\dfrac{2\pi}{3}$.

7. $\frac{1}{2}$. $1 - \tan x$. **8.** $\sec^3 a$. **9.** $-\dfrac{2c^3}{\sqrt{x_1^2 + y_1^2}}$.

11. An equal parabola. **12.** $\cos x(2-x^2) + 2x\sin x$. $\dfrac{e^x(\sin x - \cos x)}{2}$.

13. $\log\dfrac{\sqrt{(x+1)(x+3)}}{x+2}$. $\log(\log x) + \log x$. **14.** $2\pi ac$.

STATICS.

1. $\frac{2}{3}$. **3.** 10 lbs. acting towards centre of hexagon.

7. $\left(\frac{b}{a}\right)^n$: $\frac{W}{l}\left\{r_1 + \frac{2nq(r_2 - r_1)}{\sqrt{h^2 + (r_2 - r_1)^2}}\right\}$: $n\pi W\left\{2r_1 + \frac{2nq(r_2 - r_1)}{\sqrt{h^2 + (r_2 - r_1)^2}}\right\}$.

8. If α, β be the angles of smooth and rough planes,

$$\sin\alpha = \sin\beta + \mu\cos\beta.$$

If a = length of rod, c = distance of its centre of gravity from lower end, m = coefficient of friction,

$$\theta \text{ lies between 0 and } \tan^{-1}\left\{\frac{2c(m^2+1)}{a(m^2-1)} - \frac{m+1}{m-1}\right\}.$$

9. At intersection of the straight lines joining centre of escribed circles to opposite angular points of triangle.

DYNAMICS.

1. $\frac{V}{g}$. $7\frac{1}{2}''$. **3.** 80 feet a second.

4. If h, k be co-ordinates of given mark, $\alpha = \tan^{-1}\frac{2k}{h}$. $u = \frac{\sqrt{2gk}}{\sin\alpha}$.

5. 28 lbs. **7.** 3. $\frac{1}{2}$. **8.** 2. **9.** 24 feet per second.
12. 9680 feet.

SET VII.

ALGEBRA.

1. $x^3 + x(ab + bc + ca - a^2 - b^2 - c^2) + \sqrt{a-b}\,.\,\sqrt{b-c}\,.\,\sqrt{c-a}$.
$x^3 - 3x(a-b)^2 - 2(a-b)^3$. **2.** $x - 2a$. $2a$.

3. $(x + 2y + 3z)(x + 2y - 3z)(x - 2y + 3z)(2y + 3z - x)$. **1.**

4. (1). 25. (2). 0, 4; $\frac{3}{5}$, 3: (3). 35, -25; 35, -49.

5. 75 shares at £25 each. **6.** $-3 + 4\sqrt{-1}$.

7. $p^3 - 3pq$. q^3. $x^2 - 35x + 216 = 0$. **8.** 6.

10. 3, 4, 6. **11.** 80. **12.** 9.

TRIGONOMETRY.

1. 30°, 60°, 90°. **2.** $\frac{1}{\sqrt{2}}$. **6.** $n\pi + \frac{\pi}{3}$.

10. 70° 53′ 40″. 49° 6′ 20″.

PURE MATHEMATICS (1).

1. $\tan^{-1}\frac{1}{\sqrt{2}}$. **3.** $R\sin\frac{\text{arc}}{R}$.

7. $\frac{BC' - B'C}{AB' - A'B}, \frac{AC' - A'C}{A'B - AB'}$;
$a(BC' - B'C) + b(CA' - A'C) + c(AB' - A'B) = 0.$

8. $y - k = \frac{b}{a}(x - h)$.

9. 2; $\frac{5}{2}$, $-\frac{3}{2}$. $r^2 - 2r\cos\theta . \sqrt{\frac{17}{2}} + \frac{9}{2} = 0$. $x^2 + y^2 - 2\sqrt{2}(x+y) = 0$.

10. $\{x(x-a) + y(y-b)\}^2 = (a^2 + b^2)(x^2 + y^2)$.

Pure Mathematics (2).

4. $b(b-a)$. **6.** 21. 35.

7. $\frac{\log \frac{an^2 - b}{n^2 - m^2}}{\log m} : \frac{\log \frac{am^2 - b}{m^2 - n^2}}{\log n}$.

8. $\frac{n\pi}{3}$ or $n\pi + \frac{\pi}{2}$ or $2n\pi \pm \cos^{-1}\sqrt{\frac{2}{3}}$. **9.** $\frac{1}{2} \cdot \frac{s}{S} abc$.

10. $z^2(a + b - y - e) = (x - e)(a - y)(y - b)$.

12. $\frac{x}{y} = \frac{m^3 - a^3}{m^3 + b^3}$, $\frac{y}{z} = \frac{m^3 - b^3}{m^3 + c^3}$, where $m^3 = \sqrt{-\frac{a^3b^3c^3}{a^3 + b^3 + c^3}}$.

Pure Mathematics (3).

3. 1, 3, 5. $-2, \frac{1 \pm \sqrt{-3}}{2}$.

5. $x^{\tan^{-1}x}\left\{\frac{\log x}{1 + n^2} + \frac{\tan^{-1}x}{x}\right\} : e^x\left(1 + \frac{d}{dx}\right)^r x^n$.

6. 0. **9.** $x - \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} - \ldots + \frac{(-1)^{n-1}x^n}{\underline{|n}}$.

13. $\frac{2}{3}a^2(2 + \sqrt{5})^{\frac{1}{2}} + a^2\log(2 + \sqrt{5} + \sqrt{8 + 4\sqrt{5}})$.

Statics.

1. $\frac{7}{20}$ cubic feet. **2.** 10 lbs. **3.** Centre of gravity of triangle.
5. 10. **6.** 1 foot. **7.** W $(\sin\alpha + \mu\cos\alpha)$. **9.** 7763.

10. $\frac{r}{\sqrt{2}}$.

Dynamics.

1. $\frac{n^2 f}{m}$. **2.** 196 feet. 112 feet per second. **8.** $v\left(\frac{2}{3}\right)^{n-1}$.

9. When it has passed over a vertical distance $= \frac{r}{3}$. $v = \sqrt{\frac{2gr}{3}}$.

11. A diminution of $\frac{3}{4}$ of its weight.
12. $4\,el\sin^2\alpha\cos\alpha$, l being length of plane.
13. $\frac{g}{\pi^2}\cdot\frac{n(2a-n)}{(a-n)^2}$, where $a = 86400$.

SET VIII.

Algebra.

1. $1+2x-7x^2-16x^3$. **2.** $(x-a)(x-b)$. **4.** $(2x-1)(3x+1)(4x-3)$.
5. (1). 100. (2). 6, $2\frac{10}{11}$. (3). ± 3, ± 4, ± 6.
6. $8-\sqrt{15}$. $\frac{1}{\sqrt{2}}(\sqrt{15}-1)$. **7.** 0, 2, -3. 16, 2, 130.
8. 20 and 30 miles an hour. 15 and 10 hours. **9.** 5.
11. 2. 13, 15, 17, 19.

Trigonometry.

1. 128°. **2.** $\sin 2670° = \frac{1}{2}$. **10.** 9·7565636.

Pure Mathematics (1).

3. When the two straight lines are at right angles to one another.
8. $\dfrac{\sin\frac{n+1}{2}a\sin\frac{na}{2}}{\sin\frac{a}{2}}$. **9.** $\dfrac{b\cot a-a}{b+a\cot a}$.
10. $\dfrac{c^2(a_1b_2-b_1a_2)}{2(a_2b_3-a_3b_2)(a_3b_1-a_1b_3)}$.

Pure Mathematics (2).

5. 22. **6.** $\frac{441}{1000}$, $\frac{21}{40}$. **8.** ± 3.

Pure Mathematics (3).

1. $3x^3+29x+126$. 490. **2.** $-1\pm\sqrt{2}$. $-1-\sqrt{-1}$.
3. $x^3-8x^2+19x-15=0$.
4. $\dfrac{1+6x}{1+x^2+6x^3+9x^4}$. $e^{2nx}\left(\dfrac{1-nx}{1+nx}\right)^{\frac{1}{2}}\times\dfrac{n-2n^2x^2}{1-n^2x^2}$.
5. $n\pi+\dfrac{\pi}{4}+\dfrac{x}{2}-\dfrac{x^3}{12}+\dfrac{x^5}{48}$.
6. $x^m\,.\,m^n\sin\left(mx+\dfrac{n\pi}{2}\right)+n\,.\,mx^{m-1}\sin\left(mx+\dfrac{(n-1)\pi}{2}\right)+\ldots$
7. A chord inclined to axis of x at $\tan^{-1}\dfrac{b}{a}$. **9.** $\dfrac{2a^2}{b}$.

12. (1). $\frac{1}{a}\tan^{-1}\frac{x}{a}$. (2). $\frac{3}{2(1-x)}+\log\frac{(1+x^2)^{\frac{1}{2}}}{\sqrt{x-1}}-\tan^{-1}x$.

(3). $\frac{e^{ax}(a\cos nx+n\sin nx)}{a^2+n^2}$.

13. $\frac{\lfloor 2n}{2^n(\lfloor n)^2}\cdot\frac{\pi}{2}$. **14.** $\frac{4}{3}ab$.

STATICS.

1. $\frac{Mg\sin\phi}{\sin(\theta+\phi)}$. $\frac{Mg\sin\theta}{\sin(\theta+\phi)}$. **2.** 2 lbs.

3. $\frac{W\sin^2\frac{\theta}{2}}{\cos\theta}$, where θ is acute angle of rhombus.

4. Distance of centre of gravity from centre of square is $\frac{\pi(5\sqrt{2}-7)}{\sqrt{2}\{2-\pi(3-2\sqrt{2})\}}$. a, where a is side of square.

5. $2P\sqrt{2}$, and acts at an angle of 75° with lever.

8. 49 lbs. 1 lb. each.

DYNAMICS.

1. 2 minutes. **2.** 20 miles an hour. **3.** 8, $\frac{11}{15}$.

5. 6 or 2. 256 feet. **6.** g. Velocity is instantaneously changed to $\frac{3}{4}g$, after which there is no change.

SET IX.

ALGEBRA.

1. 1. **3.** $(x+y)^2+4(x+y)+16$. **4.** $3a+b+2c+d$.

5. $x-a$, $\frac{3x-a-2b}{x^2-x(a+b)+ab}$. **6.** (1). $2\frac{1}{7}$. (2). $-\frac{7}{2}\frac{7}{2}$, 3. (3). 8, 3; 2, 7.

7. 1125 and 500 feet per second.

9. $1, \sqrt{-1}, -1, -\sqrt{-1}$. $\frac{2b(3a^2-b^2)\sqrt{-1}}{a^2+b^2}$. $2a(1+\sqrt{-1})$.

11. $(p+1)^{\text{th}}$, where p is the integral part of $\frac{2a+b}{b}$. 4 or 13.

14. $\frac{5\,.\,11\,.\,13}{2^{16}}$. 10·00999.

TRIGONOMETRY.

1. $8\frac{1}{3}$. **2.** 1, −1. **3.** $\cos 4A-\cos 14A$. **4.** $n\pi-\frac{\pi}{4}$.

6. $262\frac{1}{2}$ feet. **7.** 26° 33′ 55″. 63° 26′ 5″. **8.** 1034·27517.

11. $2\frac{2}{3}$ feet.

Pure Mathematics (1).

4. If the four points are in one plane, but not inscribable in a circle, no sphere possible. If they are in one plane on the circumference of a circle, sphere is indeterminate.

5. Plane is parallel to axis of cone.

14. (1). A parabola whose L R is $\frac{1}{\sqrt{2}}$. (2). Lemniscate. (3). Hyperbola of eccentricity 2.

Pure Mathematics (2).

2. 128. **3.** 9. **4.** $\frac{1}{12}$. **9.** $5(x^2+y^2+z^2+xy+yz+xz)$.

13. 142857. **14.** (α). $\frac{\cos 2n\theta}{2\sin\theta}$. ($\beta$). $\frac{1}{2}\left\{\frac{\sin 2n\theta}{2\sin\theta}+(n-1)\cos\theta\right\}$.

15. 210.

Pure Mathematics (3).

1. 2, 2. $-2\pm\sqrt{2}$. **2.** $\frac{1}{2}$, $-\frac{2}{3}$. **4.**

4. $\frac{1}{\log_e a}$. **1.** $y^{x^2}.x^2\left\{\frac{dy}{dx}+\log y\left(\frac{2}{x}+\log z\,.\,\frac{dz}{dx}\right)\right\}$.

6. 2. ∞. 1. **10.** $\frac{2\sqrt{2}}{3}\sqrt{ar}$. **11.** $y^2=4ax$.

12. (1). $-\sqrt{2ax-x^2}+a\sin^{-1}\frac{x-a}{a}$.

(3). $\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$. (4). $\frac{1}{3}\log\frac{\sqrt{1+x^3}-1}{\sqrt{1+x^3}+1}$.

13. $\pi a^2\left(2-\frac{1}{\sqrt{2}}\right)$. $\frac{\pi a^3}{3}(2-\sqrt{2})$, a being slant side of cone.

Statics.

2. An equal force P. **4.** $1\frac{3}{4}$ feet. Yes.

5. (1). Centre of gravity of triangle. (2). Centre of inscribed circle.

9. No advantage is gained. **10.** $33\times66\times108\times112$. 4·4352.

Dynamics.

2. (1). When points coincide. $3v$.

(2). When points are at opposite ends of a diameter. O.

4. $9\frac{3}{8}''$. **5.** 120 feet per second. **11.** 5·59796. 29·31947.

14. If θ is inclination of string to vertical, $\theta=\frac{\pi}{3}$. **15.** 28521504000.

SET X.

Algebra.

2. $\frac{\sqrt{m}(m+3n)}{4}$. **3.** $b^3x^3(m+1)+abx(m+1)(n+1)+a^3(n+1)$.

4. $\frac{3x^2+7x-12}{(x^2-16)(x^2-9)}$. $\frac{1}{(x-4)(x-3)}$. **5.** $2x^2+5xy+3y^2$.

6. ·467. **7.** (1). $2\frac{1}{4}$. (2). 9, − 3. (3). $b, \frac{b}{\sqrt[3]{5}}$; $a, -\frac{a}{\sqrt[3]{5}}$. **9.** 36, 12.

12. $\frac{3}{4}, \frac{1}{4}, \frac{1}{12}\ldots$. **13.** 658. **14.** $\frac{n.(n+1)\ldots.(n+m-1)}{\lfloor m}$.

Trigonometry.

1. 120°. 108°. **2.** $\cos(4A-B)\cos(3B-A)$. **6.** $\frac{\pi}{5}$.

7. $200(\sqrt{3}-1)$ feet. **8.** 90° or 30°. 60° or 120°. 200 or 100 feet.
9. ·54. **11.** 9·9813707.

Pure Mathematics (1).

9. $\frac{ah+\beta k+\gamma}{\sqrt{a^2+\beta^2}}$. **10.** $\frac{ax+by+c}{\sqrt{a^2+b^2}} \pm \frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = 0$.

11. $-\frac{a}{2}, -\frac{b}{2}$. $\frac{\sqrt{a^2+b^2-4c}}{2}$.

12. $y\sqrt{a^2-(b-c)^2} = (b-c)x+ca$. **14.** $y = \pm\frac{bx}{a}$.

Pure Mathematics (2).

1. 0. 10. **2.** £7. 9*s*. 10*d*. **3.** $p+1 : q+1 : r+1$.
4. $1+x(a+b)+x^2(a^2+ab+b^2)+x^3(a^3+a^2b+ab^2+b^3)+\ldots$
$\frac{1}{a-b}\left\{\frac{a(1-a^nx^n)}{1-ax}-\frac{b(1-b^nx^n)}{1-bx}\right\}$.
5. $\frac{16}{5\sqrt{5}}$. **6.** 12 and 16, or 48 and 4.
7. $\frac{n(3n+1)}{2}$. $n(3n^2+6n+1)$. $\log\left(\frac{1}{2\sin\frac{\theta}{2}}\right)$.

8. $\alpha = n\pi$ or $\frac{1}{2}\left(n\pi+\frac{\pi}{2}\right)$.

14. $x+y\sqrt{3}-(6+2\sqrt{3}) \pm \{x-y\sqrt{3}-(6-2\sqrt{3})\} = 0$.

Pure Mathematics (3).

2. 1, $\frac{1}{2}, \frac{1}{4}$. **4.** $\frac{1}{2\sqrt{x-x^2}}$.

5. $y = n\pi + \frac{\pi}{4} - \frac{x^2}{4} - \frac{x^4}{\lfloor 4} + \frac{19x^6}{\lfloor 6} + \ldots$ **7.** In ratio of $\pi : 4$.

8. $y = x - \frac{a}{2} = -x + \frac{a}{2}$ and $x - a = 0$.

12. (2). $-\frac{1}{42(x-4)} - \frac{13}{1764}\log(x-4) + \frac{1}{36}\log(x+2) - \frac{1}{49}\log(x+3)$.

(3). $x \log(\sin x)$. **13.** $\frac{8a^2}{3}$.

Statics.

1. 60°. **2.** 12 and 8 lbs. **9.** $AP = \frac{\sqrt{3}-1}{2} AD$.

10. 5·6 per cent., taking a cubic foot of water to weigh 1000 ozs.

Dynamics.

1. $\frac{11}{12800}$ feet per second. **2.** (1). 0. (2). $44\sqrt{3}$ feet per second.
5. (1). Straight line. (2). Parabola. 52·8 feet from point vertically underneath balloon from which stone was dropped.
8. W cos α. **12.** (1). $60\frac{1}{2}$ ozs. $30\frac{1}{2}$ ozs.
13. $1\frac{57}{64}$. **14.** 1·02″ nearly. **15.** 432.

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